## 3-13.1 Rotation of Objects in Space and Antenna Positioners

To use the matrix to rotate an object in space, pre-multiply the rotation matrix by the position vector (vector on the left and matrix on the right). The only way the matrix multiplication makes sense is to use a row position vector.

In the second method for figuring out the rotation matrix work from left to right with the three single-axis rotation matrices. In this case the axes do not change direction. First rotate about the X-axis, the Y- and Z-axes remain in their normal orientation. A second rotation about the Y-axis, for example, is still about the real Y-axis in the reference coordinate system. Rotation matrices to the right rotate not only the point in space when it pre-multiplies the overall rotation matrix, but the rotation matrices to the left. For example, if we use two rotation-matrices [Y][Z] where [Y] is a rotation about the Y-axis, then the rotation about the Z-axis [Z] has rotated the Y-axis rotation when the matrices are multiplied in this order. The order of rotation matrices is the same as describing the rotation of the object in spherical coordinates: (θ, ϕ). The rotation about the Z axis is ϕ and the rotation θ is about the rotated Y-axis. We use the rotation matrix [Y][Z]. Either this method or the first one produces a proper rotation matrix, but when using **Euler** remember the rotations are multiplied from right to left. When using the second method send the rotations in reverse order along with reversing the order of the axes indication numbers.

We use products of these axis rotations to reorient an object or pointing direction. Consider the rotation of a position by the product of three rotation matrices:

$$\begin{bmatrix} X_{old} & Y_{old} & Z_{old} \end{bmatrix} \mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3 = \begin{bmatrix} X_{rotated} \\ Y_{rotated} \\ Z_{rotated} \end{bmatrix}$$

The logical approach is to multiply the 3 x 3 matrices, $\mathbf{R_1}$, $\mathbf{R_2}$, and $\mathbf{R_3}$, before multiplying by the position vector. When we postmultiply $\mathbf{R_1}$ by $\mathbf{R_2}$, it rotates the axis of rotation of $\mathbf{R_1}$. The postmultiplication by $\mathbf{R_3}$ rotates the rotation axis $\mathbf{R_2}$ and $\mathbf{R_1}$ is rotated once more. We can take the rotations one by one from left to right and use the rotation matrices about each of the principal axes provided we convert the column vector back to a row vector after each multiplication.

A convenient way to define the orientation of objects in space is to use spherical coordinate angles, since they are the same as pattern angles. We line up the matrices from right to left in this case. When rotating the coordinate system about an axis, the other axes change direction. The next rotations are about these new axes. The three rotations are often called the *Euler angles*. We use the following three rotations for the spherical coordinate pointing:

1) *Z*-axis rotation = ϕ
2) New *Y*-axis rotation = θ
3) New *Z*-axis rotation: aligns the polarization of the antenna

The last rotation takes some thought because the first two rotations have altered the orientation of the antenna.

When calculating the pattern of the array for a particular direction, first compute rectangular components of the direction vector and the two polarization vectors. Multiply the direction vector by $k$ (2π/λ) and take the dot (scalar) product with the position vector to calculate phasing of a particular element. You need to determine the pattern direction in the rotated antenna's coordinate system found by using Eq. (3-31). Multiply the rotation matrix by the unit direction vector placed to the right. When you convert the output vector to spherical coordinates, you obtain pattern coordinates of the rotated antenna. Both the pattern components of the rotated element and the unit polarization vectors are needed. In the next operation you rotate the prime coordinate polarization unit vectors into the rotated element coordinate system using the same operation as the direction vector.
You calculate final radiated components by projecting the rotated prime coordinate polarization vectors on the element pattern unit polarization vectors.

$$\begin{aligned} E_\theta &= E_{\theta,element}\hat{\theta}_{element}\cdot\hat{\theta}_{rotated} + E_{\phi,element}\hat{\phi}_{element}\cdot\hat{\theta}_{rotated} \\ E_\phi &= E_{\theta,element}\hat{\theta}_{element}\cdot\hat{\phi}_{rotated} + E_{\phi,element}\hat{\phi}_{element}\cdot\hat{\phi}_{rotated} \end{aligned} \tag{3-34}$$

Since we measure element patterns on antenna positioners, it is convenient to consider positioners as a series of coordinate system rotations.

## Rotation about an Arbitrary Axis

In some cases we need to rotate an object about an arbitrary axis which is not one of the principal axes. This case requires a little more work, but if we translate the object before and after rotation, we can find a single rotation matrix. By using the ends of the axis given by the two position vectors: $a_1$ and $a_2$, we use $a_1$ as the origin and compute the rotation angle by using a counter-clockwise rotation around the axis from $a_1$ to $a_2$. Find the vector $a_2 - a_1$ and convert it to spherical coordinates ($\theta$, $\phi$). Find the rotation matrix which would rotate the direction vector from the Z-axis to the rotation axis. The first rotation is about the Z-axis by $\phi$ and the second is about the new Y-axis by $\theta$. The three Euler angles are: 1) $\phi$ about Z-axis, 2) $\theta$ about the Y-axis, 3) zero about Z-axis which gives a rotation matrix R. The inverse of the rotation matrix will rotate an object in the opposite direction. A rotation matrix is orthogonal which means its transpose is its inverse. We rotate the axis of rotation to the Z-axis, rotate about the Z-axis, and then rotate back to the original coordinate system. Translate a position vector by subtracting the vector $a_1$, post-multiply by the rotation matrix given by $R^T R_Z R$, where $R_Z$ is the Z-axis rotation about the axis, and then add the translation vector $a_1$ to the rotated position vector.

No matter which method you use to find the rotation matrix, the multiplication of the rotation matrix and the vector, place the position (direction) vector on the right of the matrix if you need the vector expressed in the rotated coordinate system. To rotate an object, place the vector on the left of the rotation matrix as a row vector and perform the multiplication.

## Model Tower Positioner

We can describe antenna positioners in terms of rotations about axes in space. Consider a model tower positioner mounted on top of an azimuth table. This positioner moves the antenna exactly in spherical coordinates with $\theta$ corresponding to the azimuth axis and $\phi$ corresponding to the position axis on top of the model tower. A horizontally polarized source antenna measures the $\theta$ polarization and a vertically polarized source measures the $\phi$ polarization. The horizontal plane contains the X and Z axes. Vertical is the Y-axis. The positioner rotates the antenna about a fixed Y-axis (azimuth angle) and changing direction Z-axis (head axis). Because the Z-axis rotation is rotated by the Y-axis rotation, the rotation matrices are ordered [Z][Y].

$$\begin{bmatrix} \cos H & -\sin H & 0 \\ \sin H & \cos H & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos A & 0 & \sin A \\ 0 & 1 & 0 \\ -\sin A & 0 & \cos A \end{bmatrix}$$

Positive rotations of the axes moves the axis clockwise. Hence, the angles A (azimuth) and H (AUT or head) are negative and reflected in the expression above. When the matrices are multiplied, we get the expression

$$\begin{bmatrix} \cos H \cos A & -\sin H & \cos H \sin A \\ \sin H \cos A & \cos H & \sin H \sin A \\ -\sin A & 0 & \cos A \end{bmatrix} = \begin{bmatrix} \text{Rotated X - axis} \\ \text{Rotated Y - axis} \\ \text{Rotated Z - axis} \end{bmatrix}$$

Each row of the matrix is the location of one of the axes on a unit sphere. The positioner rotates the antenna which has been represented as a unit radius sphere. We need to know what direction ($\theta$,$\phi$) on the rotated unit sphere is pointed along the Z-axis, the direction of the source antenna. The projection along the Z-axis of each rotated axis is given by the dot product of the vector with the unit vector along the Z-axis.

$$\text{Z - axis direction of sphere after rotation} = \begin{bmatrix} \cos H \sin A \\ \sin H \sin A \\ \cos A \end{bmatrix} = \begin{bmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{bmatrix}$$

The second vector shows that rotation of the head or AUT axis corresponds to a $\phi$ rotation and the azimuth table rotation corresponds to the $\theta$ rotation in a spherical coordinate system.

### Azimuth over Elevation Positioner

A common heavy positioner is the azimuth over elevation positioner where the source direction is usually along the horizontal plane. The boresight axis (Z-axis) of the antenna is mounted 90° from the vertical Y-axis of the azimuth table when elevation is zero. The elevation axis is a horizontal X-axis. Positive rotation of the positioner is clockwise. Since the Y-axis (azimuth) changes direction when the elevation axis rotates, the rotation matrices are ordered [Y][X] or

$$\begin{bmatrix} \cos A & 0 & \sin A \\ 0 & 1 & 0 \\ -\sin A & 0 & \cos A \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos E & -\sin E \\ 0 & \sin E & \cos E \end{bmatrix} = \begin{bmatrix} \cos A & \sin A \sin E & \sin A \cos E \\ 0 & \cos E & -\sin E \\ -\sin A & \cos A \sin E & \cos A \cos E \end{bmatrix} = \begin{bmatrix} X-axis \\ Y-axis \\ Z-axis \end{bmatrix}$$

In the same manner as the model tower we represent the antenna on the positioner as a rotated unit sphere. The point on the sphere lying along the Z-axis and pointed towards the source antenna is found by projecting each axis along the Z-axis that gives

$$\begin{bmatrix} \sin A \cos E \\ -\sin E \\ \cos A \cos E \end{bmatrix} = \begin{bmatrix} R_X \\ R_Y \\ R_Z \end{bmatrix}$$

The normal pattern direction can be found by converting the unit vector to spherical coordinates.

$$\theta = \cos^{-1}(R_Z) = \cos^{-1}(\cos A \cos E) \text{ and } \phi = \tan^{-1}\left(\frac{R_Y}{R_X}\right) = \tan^{-1}\left(\frac{-\sin E}{\sin A \cos E}\right)$$

### Elevation over Azimuth Positioner

This positioner has the elevation axis above the azimuth and changes direction when the azimuth moves. In the zero position the elevation axis is the X-axis and the azimuth axis is the Y-axis when the boresight of the range is along the Z-axis. The positioner has the rotation matrix [X][Y] or

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos E & -\sin E \\ 0 & \sin E & \cos E \end{bmatrix} \begin{bmatrix} \cos A & 0 & \sin A \\ 0 & 1 & 0 \\ -\sin A & 0 & \cos A \end{bmatrix} = \begin{bmatrix} \cos A & 0 & \sin A \\ \sin E \sin A & \cos E & -\sin E \cos A \\ -\cos E \sin A & \sin E & \cos E \cos A \end{bmatrix}$$

In the same manner as the model tower we represent the antenna on the positioner as a rotated unit sphere. The point on the sphere lying along the Z-axis and pointed towards the source antenna is found by projecting each axis along the Z-axis that gives

$$\begin{bmatrix} \sin A \\ -\sin E \cos A \\ \cos E \cos A \end{bmatrix} = \begin{bmatrix} R_X \\ R_Y \\ R_Z \end{bmatrix}$$

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