RADIATION FROM APERTURES

When analyzing large antennas, that is, large in terms of wavelengths, many of these antennas can be considered as apertures. They are analyzed with a common aperture theory. The method is exact in some cases but will only give approximate answers or valid results only over part of the pattern for most cases. The simplicity of the method is its best virtue. We have already used the method for horns and slots. It may be applied to planar arrays approximately and helps to give bounds on the pattern. The second consideration is that we usually separate out the polarization effects and are left with a scalar problem.

We can consider apertures with various distributions of fields and find the radiation pattern by a Fourier transform relation for the far field. This will involve, for a planar aperture, a two dimensional transform, but in many cases the distribution can be separated into X and Y components as a product relation. When this is done, then the transform separates into a product of transforms and we can consider single dimensional Fourier transforms. We are familiar with the one dimensional Fourier transform from signal processing which helps us visualize the pattern in various planes from the size and distribution in the aperture plane. Large apertures give small beams similiar to long time intervals which give low frequency responses. The sidelobes of the pattern are related to the harmonics of the equivalent time waveforms under Fourier transform. Rapid transitions in the time response leads to high harmonics in the frequency domain (Fourier transform). Rapid transitions in the amplitude in the aperture plane gives high sidelobes (harmonics) in the far field response (Fourier transform). This is convenient when both areas use the same mathematical development.

We need to look at the problem with some rigor before we simplify the analysis so that we can go backwards and fill it in when needed. On page 211 the induction theorem was developed to relate equivalent magnetic and electric surface currents to the electric and magnetic fields, respectively, in the aperture. We performed a similiar operation when we analyzed a slot in an infinite ground plane by the equivalence theorem where we only have a magnetic surface current. In the case of the slot we used image theory and obtained a solution valid only on one side of the infinite plane. The equivalence theorem gave us an exact solution. When we use the induction theorem, we obtain only approximate solutions because the ground screen is still present and usually ignored. It limits us to accurate solutions normal to the screen, that is, not too close to the directions of the ground plane. We used this induction theorem solution to analyze horn patterns which do not, in general, have apertures in large ground planes. There is no distinction made between a horn in free space and a horn opening in a ground plane. Nevertheless, the method with all its approximation can accurately predict the gain of the horn.

Suppose we find the magnetic and electric equivalent surface currents from the fields in a planar aperture, S, which we will take as in the X-Y plane. From these currents we can find the electric and magnetic vector potentials.

$$\bar{F} = \iint_S \frac{\bar{M}_s e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} ds \qquad \bar{A} = \iint_S \frac{\bar{J}_s e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} ds$$

Where k has been substituted for β as the propagation constant which fits the general literature. Also the rigorous expressions for the vector potentials have been used which are valid in the near field as well as the far field. $\bar{r}$ is the vector to the field observation point and $\bar{r}'$ is the vector to the source point in the aperture. To be completely rigorous, we need the fields over an infinite aperture (a closed boundary). The fields outside of a large aperture will be nearly zero and we can integrate only over the physical aperture.

FRESNEL AND FRAUNHOFER REGIONS

These regions are characterized by the type of approximations which are made in the integrals. The Fresnel region is the closest to the aperture of the two regions and is the near far field while the Fraunhofer region corresponds to the usual far field approximation. Both approximations substitute the field (observation) distance r for $|\bar{r} - \bar{r}'|$ in the amplitude term. This means that we get for the vector potentials:

$$\bar{F} = \frac{1}{4\pi r}\iint_S \bar{M}_s e^{-jk|\bar{r}-\bar{r}'|} dxdy \qquad \bar{A} = \frac{1}{4\pi r}\iint_S \bar{J}_s e^{-jk|\bar{r}-\bar{r}'|} dxdy$$

Fresnel and Fraunhofer Regions

The only difference between the two regions is the manner in which the phase term is handled. Let us expand the phase term.

$$|\bar{r} - \bar{r}'| = (r^2 + r'^2 - 2\,\bar{r}\cdot\bar{r}')^{\frac{1}{2}}$$

Expanding this in a Taylor series in 1/r terms, we get:

$$|\bar{r} - \bar{r}'| = r - \hat{r}\cdot\bar{r}' + \frac{1}{2r}(r'^2 - (\hat{r}\cdot\bar{r}')^2) + \ldots$$

Where $\hat{r}$ is a unit vector in the direction of the field point. In the Fraunhofer approximation we retain the first two terms.

$$|\bar{r} - \bar{r}'| \approx r - \hat{r}\cdot\bar{r}' = r - r' \cos\psi$$

Where ψ is the angle between the source and field point vectors. The vector potentials become:

$$\bar{F} = \frac{e^{-jkr}}{4\pi r}\iint_S \bar{M}_s e^{jkr'\cos\psi} ds \qquad \bar{A} = \frac{e^{-jkr}}{4\pi r}\iint_S \bar{J}_s e^{jkr\cos\psi} ds$$

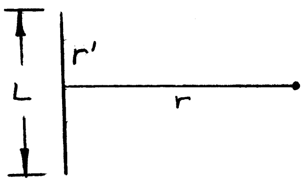
These are the usual far field expressions.

The Fresnel zone approximation retains the terms in $(r')^2$ which gives the the following integral for the electric vector potential.

$$\bar{F} = \frac{e^{-jkr}}{4\pi r}\iint_S \bar{M}\, \exp\left(k\left(\hat{r}\cdot\bar{r}' + \frac{(\hat{r}\cdot\bar{r}')^2}{2r} - \frac{r'^2}{2r}\right)\right) ds$$

A similiar expression is given for the Fresnel zone magnetic vector potential.

We have only applied these approximations to planar apertures but they hold equally for linear or volumetric distributions of sources. There is no clear boundary between the near, Fresnel, and Fraunhofer regions. It is common to the take the start of the Fraunhofer region to be somewhere between L^2/λ and $2L^2/\lambda$ where L is the maximum dimension. Let us find the phase error difference between the Fresnel and Fraunhofer approximations at these distances along the axis normal to L.



$$\hat{r}\cdot\bar{r}' = 0$$

$$r' = \frac{L}{2}$$

$$r = \frac{L^2}{\lambda} \qquad \text{Phase Error} = \frac{2\pi}{\lambda}\frac{r'^2}{2r} = \frac{\pi}{\lambda}\frac{(L/2)^2}{L^2/\lambda} = \frac{\pi}{4}$$

$$r = \frac{2L^2}{\lambda} \qquad \text{Phase Error} = \frac{\pi}{8}$$

The second is the usual phase error given for the far field approximation for antenna patterns. The Fresnel zone is usually taken as the region:

$$1 < (r/r') < (r'/\lambda)$$

Antenna patterns are usually taken at least at a distance $2L^2/\lambda$ from the antenna, but this distance may be not sufficient for low sidelobe antennas because these phase errors will raise the measured sidelobes and broaden the measured beam. We will discuss this further when we consider phase error problems of low sidelobe antennas. The far field or Fraunhofer region may be further than indicated by this bound.

We can use the vector potentials to find the fields from apertures but we will find it convenient to use the aperture fields more directly. Consider the following two integrals which are one half of Fourier transform pairs involving aperture fields and the radiation fields.

$$\bar{f} = \iint_S \bar{E}_0(x,y)\, e^{jkr'\cos\psi}\, ds$$

$$\bar{g} = \iint_S \bar{H}_0(x,y)\, e^{jkr'\cos\psi}\, ds$$

Where the Fraunhofer approximation has been used; the Fresnel approximation may be obtained by comparing to the integrals above. When we relate these integrals through vector potentials, then we can find the radiated fields in terms of them. E_o and H_o are the aperture fields which are related to the currents:

$$\overline{M}_s = \overline{E} \times \overline{a}_z \qquad \overline{J}_s = \overline{a}_z \times \overline{H}$$

Above are the surface currents for an aperture in the X-Y plane. When we expand these, we get

$$\overline{M}_s = E_y \overline{a}_x - E_x \overline{a}_y \qquad \overline{J}_s = - H_y \overline{a}_x + H_x \overline{a}_y$$

The vector potentials in terms of the integrals become:

$$\overline{F} = (f_y \overline{a}_x - f_x \overline{a}_y)\frac{e^{-jkr}}{4\pi r} \qquad \overline{A} = (-g_y \overline{a}_x + g_x \overline{a}_y)\frac{e^{-jkr}}{4\pi r}$$

In the far field we have for the electric field

$$\overline{E} = -j\omega\mu \overline{A} - j\eta\omega\epsilon \overline{F} \times \overline{a}_r$$

When we substitute in this expression the vector potentials in terms of the Fourier integrals, the electric field becomes

$$\overline{E} = \frac{j\omega\sqrt{\mu_o\epsilon_o}}{4\pi r} e^{-jkr}((g_y \overline{a}_x - g_x \overline{a}_y)\eta - (f_y \overline{a}_x - f_x \overline{a}_y) \times \overline{a}_r$$

The far field can only have θ and ϕ components which are projections of the above field on the $\overline{a}_\theta$ and $\overline{a}_\phi$ vectors (scalar vector product).

$$E_\theta = \overline{E} \cdot \overline{a}_\theta \qquad E_\phi = \overline{E} \cdot \overline{a}_\phi$$

When these products are performed, we get the results:

$$E_\theta = \frac{jke^{-jkr}}{4\pi r}(f_x \cos\phi + f_y \sin\phi + \eta\cos\theta(-g_x \sin\phi + g_y \cos\phi))$$

$$E_\phi = -j\frac{ke^{-jkr}}{4\pi r}((f_x \sin\phi - f_y \cos\phi)\cos\theta + \eta(g_x\cos\phi + g_y \sin\phi))$$

If we assume that the magnetic field is related to the electric field in the aperture as a free space plane wave then

$$\eta g_y = f_x \quad \text{and} \quad -\eta g_x = f_y$$

since

$$\eta H_y = E_x \quad \text{and} \quad -\eta H_x = E_y \quad \text{in the aperture.}$$

With this approximation the far field equations become

$$E_\theta = \frac{jke^{-jkr}}{4\pi r}(1+\cos\theta)(f_x \cos\phi + f_y \sin\phi)$$

$$E_\phi = -j\frac{ke^{-jkr}}{4\pi r}(1+\cos\theta)(f_x \sin\phi - f_y \cos\phi)$$

These are the same formulas as obtained on page 214 for the Huygens source. Of course, the same assumptions were made about the relationship of the electric and magnetic fields in the aperture for the Huygens source. We will be interested in the fields near boresight which reduces the equations to

$$E_\theta = \frac{jke^{-jkr}}{2\pi r}(f_x \cos\phi + f_y \sin\phi)$$

$$E_\phi = -j\frac{ke^{-jkr}}{2\pi r}(f_x \sin\phi - f_y \cos\phi)$$

$(1+\cos\theta) \simeq 2$ (We ignore the obliquity factor.)

We can easily eliminate polarization from these expressions by separating the polarization in the aperture. If we only have an X component of the electric field in the aperture, then there will only be a f_x component in the far field which leads to an X component of the electric field in the far field.

$$E_x = \frac{jke^{-jkr}}{2\pi r} f_x$$

The two radiation components are the projection of this field on to the a_θ and a_ϕ unit vectors. Except for the 1/R amplitude factor and the propagation phase factor, the voltage radiation intensity is proportional to f_x, the Fourier transform of the aperture field.

$$f = \iint_S E(x,y)\, e^{jk(x\sin\theta\cos\phi + y\sin\theta\sin\phi)}\,dx\,dy$$

We can define a vector propagation constant

$$k = \bar{a}_x k_x + \bar{a}_y k_y + \bar{a}_z k_z$$

where $k_x = k \sin\theta \cos\phi$, $k_y = k \sin\theta \sin\phi$, and $k_z = k \cos\theta$. We can write the Fourier integral as

$$f(k_x, k_y) = \iint_S E(x,y)\, e^{j\bar{k}\cdot\bar{r}}\,dx\,dy$$

We show f as a function of k_x and k_y which is referred to as the k space components. This is an expansion of the fields in terms of plane waves. Since the radiation of an antenna is into an unbounded region, the expansion of the field in terms of rectangular harmonics gives an infinite set of eigenvalues which are the k_x and k_y components and the sum of eigenfunctions becomes an integral. The boundary condition is the fields in the aperture plane which is satisfied by the integral of the Fourier transform over the aperture. This forms the basis for planar near field measurements. Similiarly, expansions in cylindrical and spherical eigenfunctions leads to the other two types of near field measurements.

The inverse transform can be used to find the field in the aperture plane.

$$E(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x,k_y)\, e^{-j\bar{k}\cdot\bar{r}}\, dk_x\, dk_y$$

With a given pattern in k space, we can find the aperture field. We need to know the phase as well as the amplitude of f. Usually only the amplitude is known or specified. The second problem is that the transform will give values for the aperture over an infinite aperture which will be truncated at some point and have some effect on the final pattern. This method can be used in a limited fashion to synthesize aperture distributions from patterns.

In the process of finding the transform of the aperture field we find $f(k_x,k_y)$ for all values of $\bar{k}\cdot\bar{r}'$. The maximum value of $\sin\theta = 1$ which means the range of $\bar{k}\cdot\bar{r}'$ in the visible range is limited. The values of f outside the visible range correspond to stored magnetic and electric fields in the near field of the aperture. In a synthesis procedure using the inverse Fourier transform we are free to pick the values of f outside the visible region which will simplify the aperture field.

RECTANGULAR APERTURE

We can illustrate these ideas with a rectangular aperture in the X-Y plane with a uniform field.

$$f(k_x,k_y) = \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} E_o\, e^{j(k_x x + k_y y)}\, dy\, dx$$

This integral can be separated into the product of two integrals

$$f(x,y) = E_o \int_{-a/2}^{a/2} e^{jk_x x}\, dx \int_{-b/2}^{b/2} e^{jk_y y}\, dy$$

$$f(x,y) = 4abE_o\, \frac{\sin(k_x a/2)}{k_x a/2}\, \frac{\sin(k_y b/2)}{k_y b/2}$$

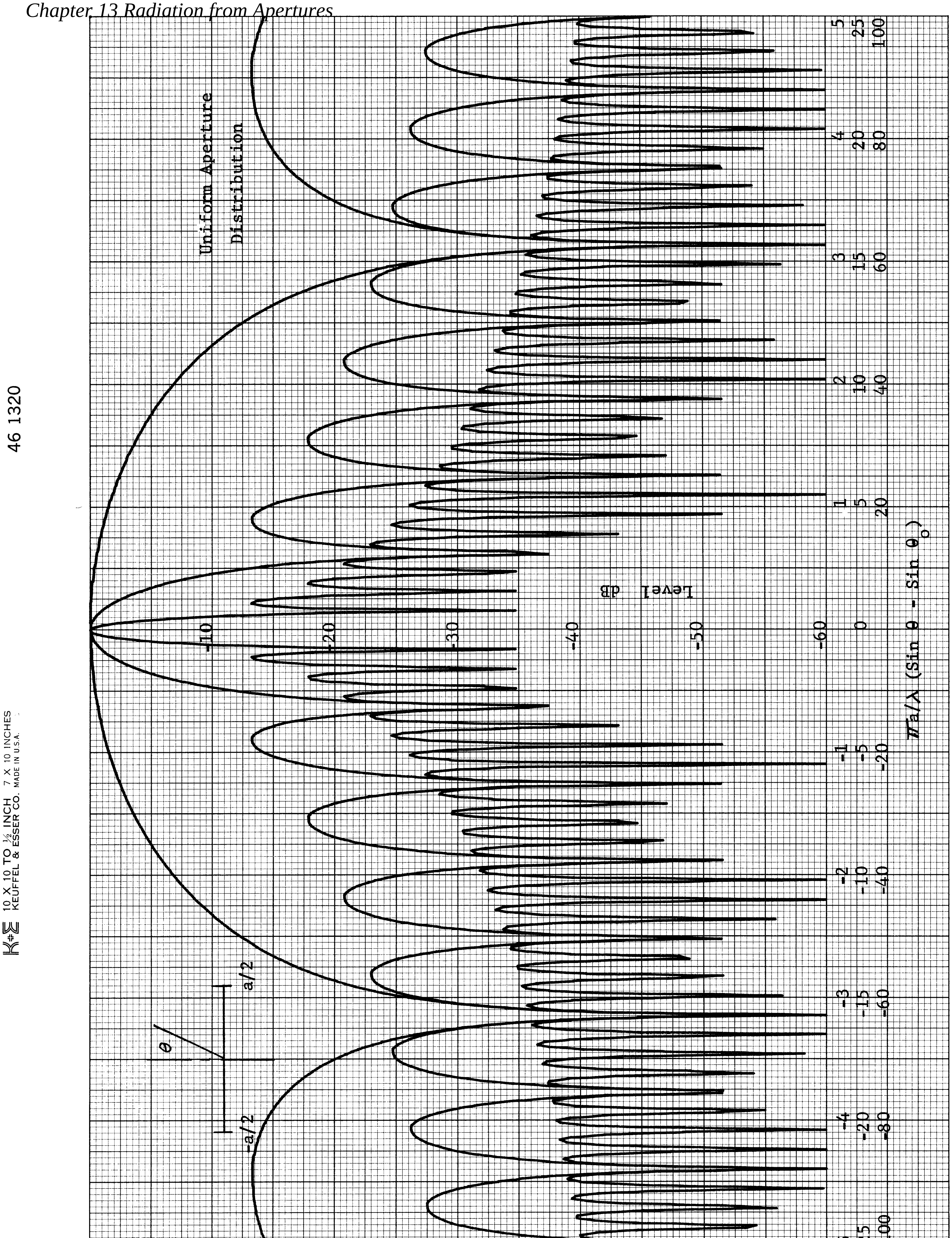
The pattern in the principle planes is found by considering k_x and k_y.

$$k_x = k \sin\theta \cos\phi \qquad k_y = k \sin\theta \sin\phi$$

$$f_{\phi=0} = 4\,a\,b\,E_o\, \frac{\sin(k_x a/2)}{k_x a/2} \qquad f_{\phi=90} = 4\,a\,b\,E_o\, \frac{\sin(k_y b/2)}{k_y b/2}$$

The pattern in each plane is determined from the aperture direction contained in that plane. This is a reiteration of pattern multiplication.

The maximum value of k_x or k_y is one in the visible region. The bounds on the visible region of $k_x a/2$ are $\mp \pi a/\lambda$. The general pattern is drawn on page 536 which is a repeat of the pattern on page 119 for the uniform line source with ka/2 as the abscissa. The values of ka/2 in the visible region are determined by the width of the aperture. Larger values of a or b of the aperture dimension extends the visible region in k space. We


Uniform Aperture
Distribution
Level dB
$\pi a/\lambda$ (Sin θ - Sin θ_o)
-10
-20
-30
-40
-50
-60
a/2
-a/2
θ
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

also see that the beamwidth decreases with increasing aperture widths. The half power beamwidths in the principle planes are given by

$$0.886\lambda/a \quad \text{and} \quad 0.886\lambda/b \quad \text{in} \quad k \text{ space.}$$

Example: Find the beamwidth of an aperture 12 wavelengths wide with a uniform distribution.

$$\text{Sin}(BW/2) = 0.886\lambda/(2a) = 0.886/24$$

$$BW = 2\ \text{Sin}^{-1}(0.0369) = 4.23^{o}$$

We could approximate the Sin(X) by X in radians for large apertures.

$$BW/2\ (\text{Radians}) = \text{Sin}(BW/2)$$

$$BW = 2\ \frac{180}{\pi}\ \frac{0.886\lambda}{2\ a} = 50.8^{o}\lambda/a$$

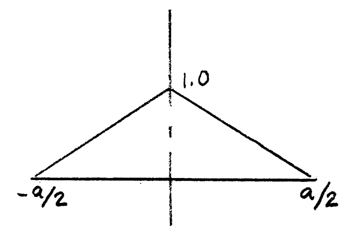
In the example this becomes: $BW = 4.23^{o}$ or the same as the exact formula.

From the plot on page 536 we can see that the first sidelobe is 13.3 dB below the main beam.

TAPERED DISTRIBUTIONS

We can find the response of tapered distributions by performing the Fourier transform or by consulting tables of transforms and making the proper substitutions to obtain the pattern in Sin θ space.

Triangular



This has the voltage distribution shown in the figure. The Fourier transform can be found in tables.

$$f(k_x) = (\text{Sin}(k_x a/4)/(k_x a/4))^2$$

A plot of this distribution is given on page 538.

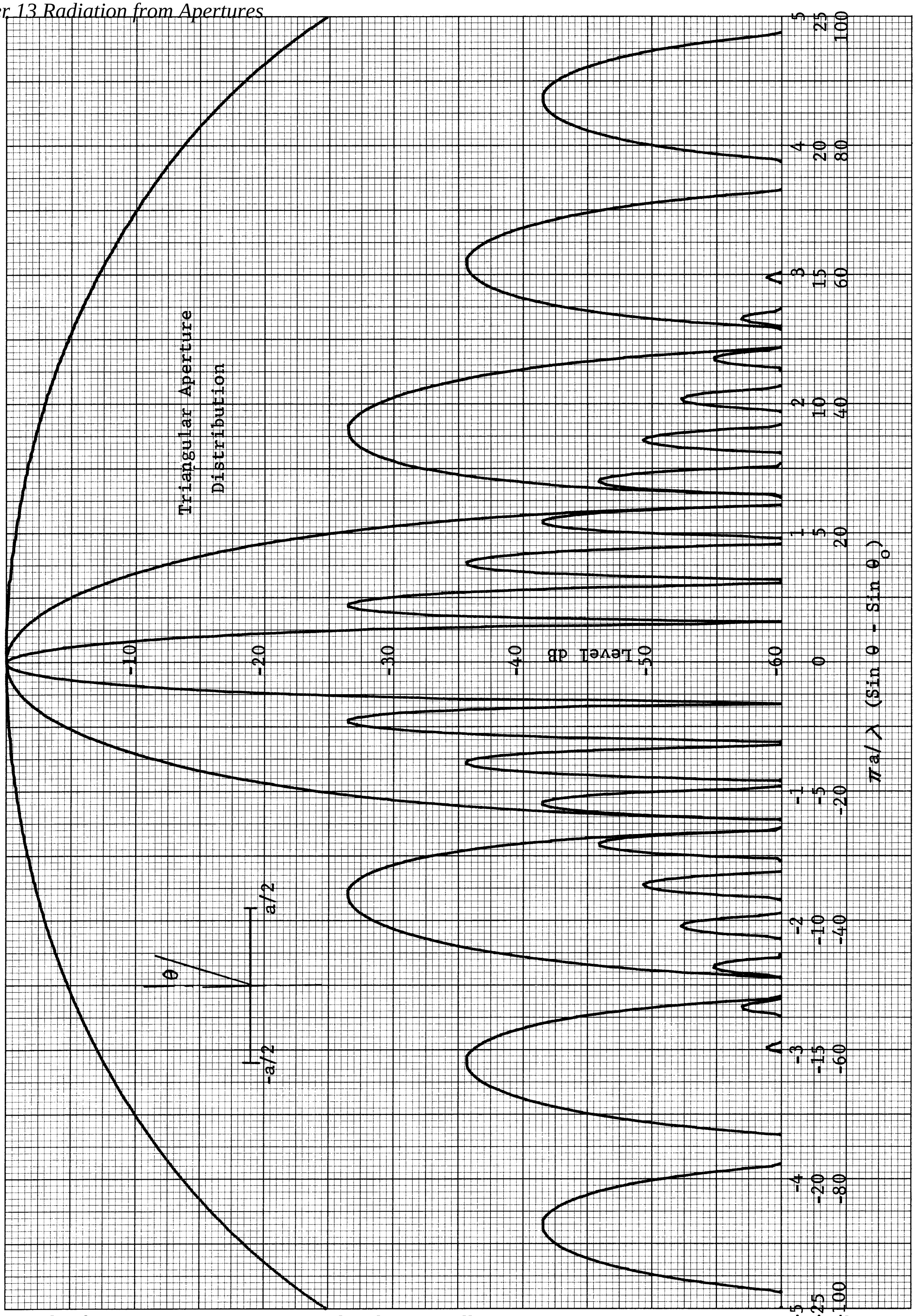
Beamwidth (in K space) = $1.275\lambda/a$

Sidelobe Level = 26.5 dB

Cosine

The cosine distribution tapers to zero on the edges and is the amplitude distribution of the H plane of a rectangular horn. The distribution is given by

$$E(x) = E_o\ \text{Cos}(\pi x/a) \qquad f(k_x) = \frac{\text{Cos}(k_x a/2)}{1 - (a/\lambda)^2}$$


Triangular Aperture
Distribution
Level dB
-10
-20
-30
-40
-50
-60
a/2
-a/2
θ
πa/λ (Sin θ - Sin θo)
-5 -4 -3 -2 -1 0 1 2 3 4 5
-25 -20 -15 -10 -5 5 10 15 20 25
-100 -80 -60 -40 -20 20 40 60 80 100

The distribution for the Cosine aperture distribution is plotted on page 540. The beamwidth and sidelobe level is read from this plot in k space.

$$\text{Beamwidth (k space)} = 1.189\,\lambda/a$$

$$\text{Sidelobe Level} = 23 \text{ dB}$$

Cosine Squared Aperture distribution

$$E(x) = E_o \cos^2(\pi x/a) \qquad f(k_x) = \frac{\sin(k_x a/2)}{k_x a/2\,(1 - (a/2\lambda)^2)}$$

The k space pattern is plotted on page 541 from which the beamwidth and first sidelobe levels are found.

$$\text{Beamwidth (k space)} = 1.44\,\lambda/a$$

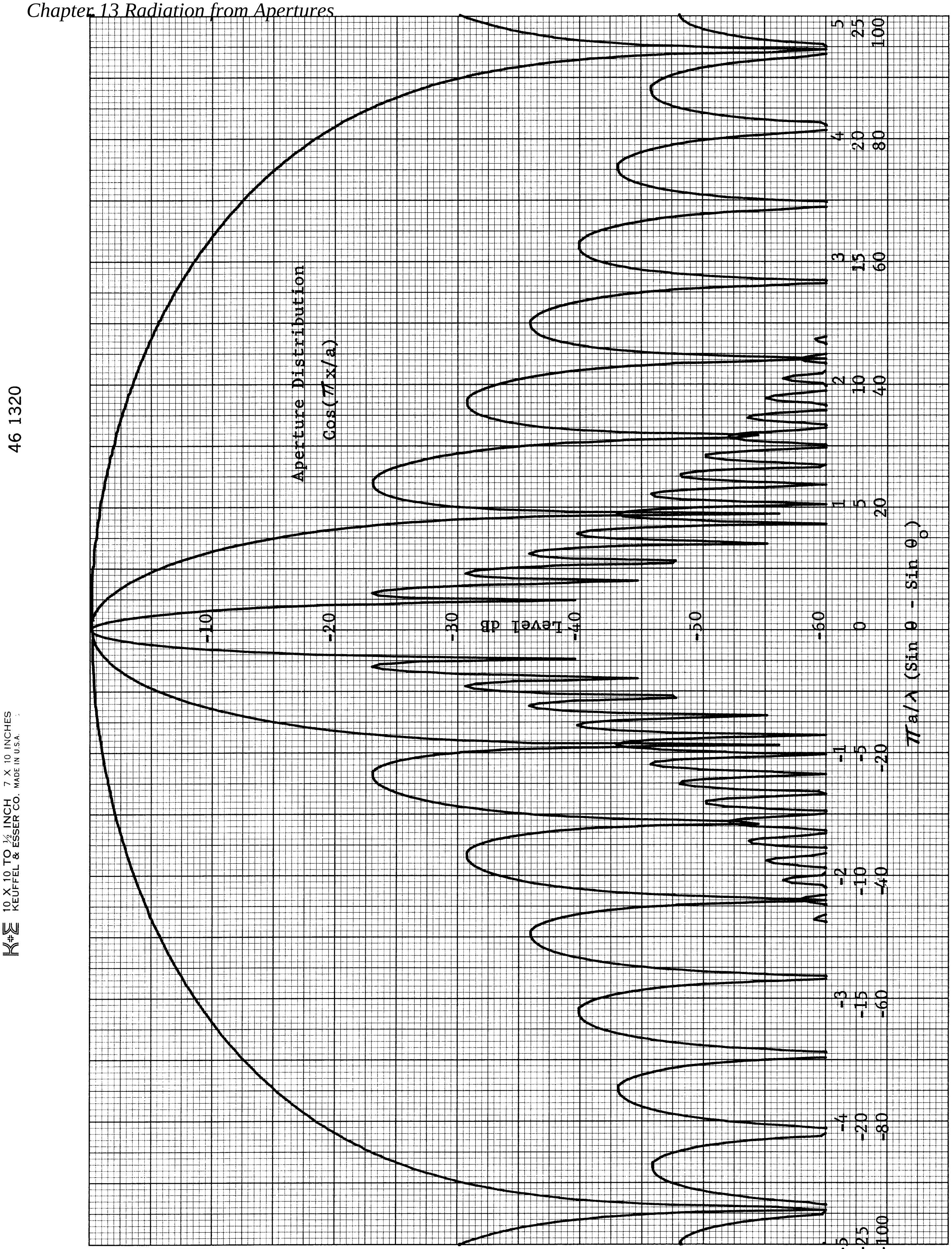
$$\text{Sidelobe Level} = 31.5 \text{ dB}$$

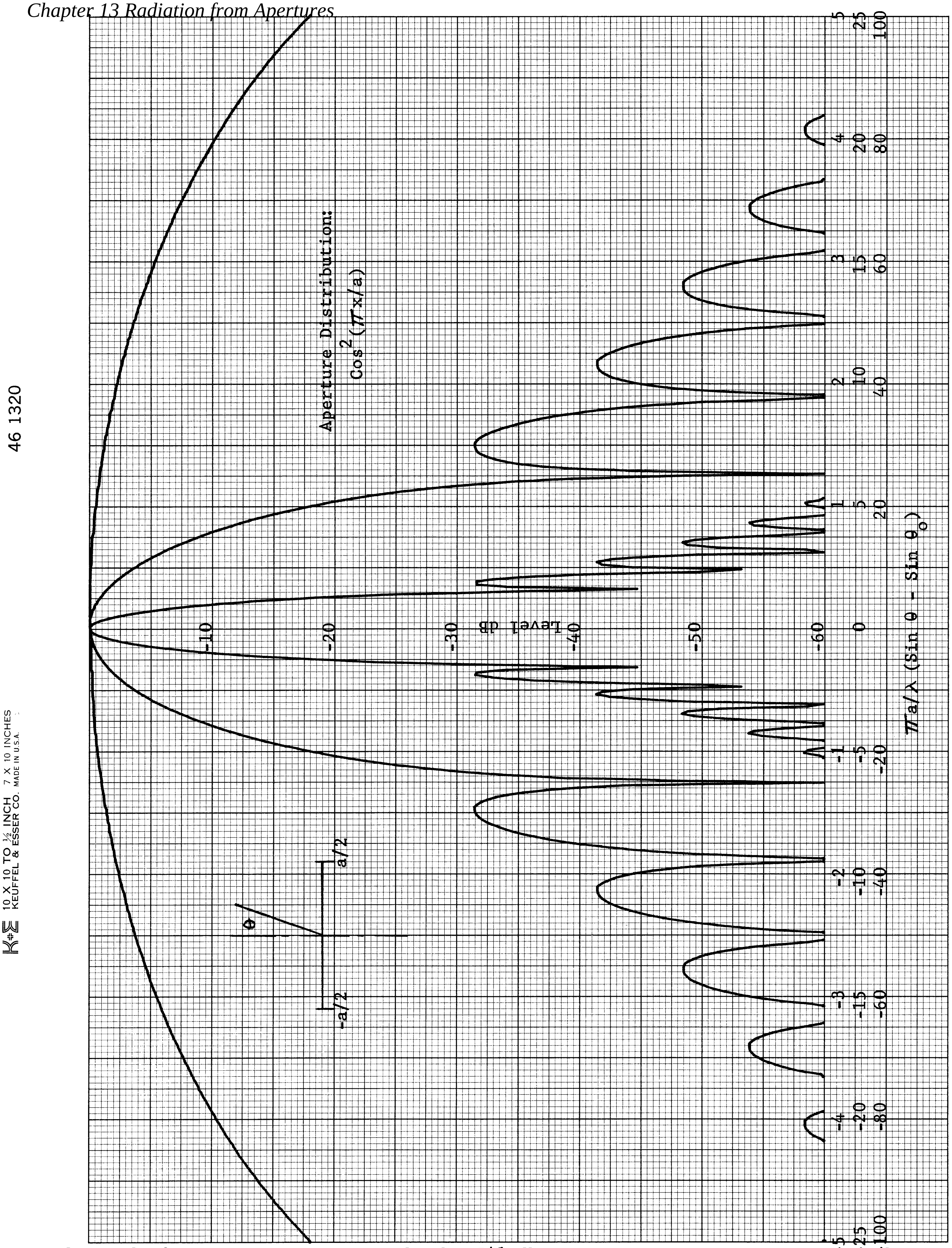
Cosine Squared on a Pedestal Aperture Distribution

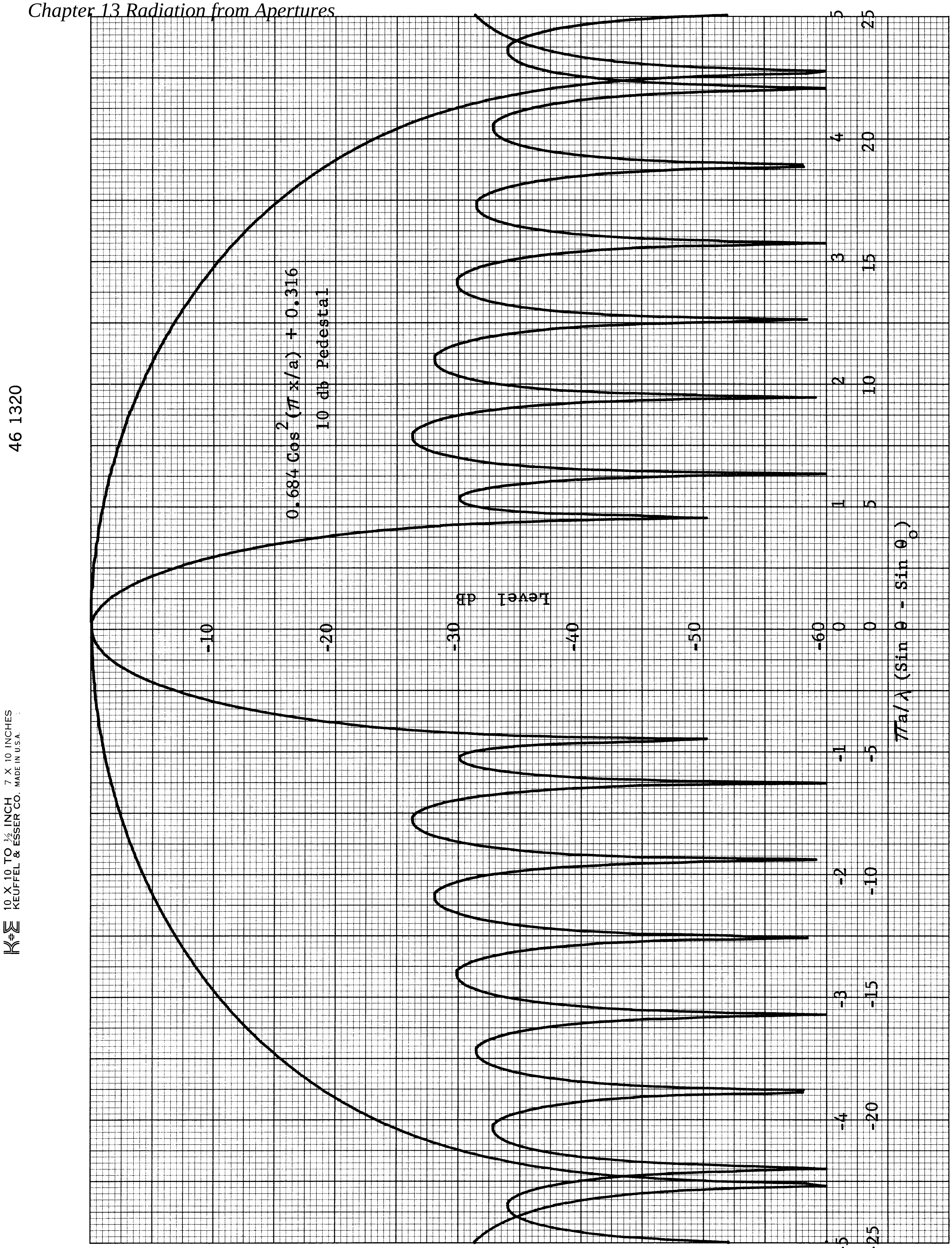
There are three advantages to raising the cosine squared aperture distribution on a pedestal. First the beamwidth can be decreased from the zero pedestal distribution. Second the maximum sidelobe can be reduced because the sum of the two portions of the pattern due to the pedestal and the cosine square distributions will cancel the near sidelobes. Comparing the plots on page 536 for the uniform distribution and the plots on page 541 shows that the first sidelobes of the uniform distribution are within the main beam of the cosine squared distribution k space pattern. The sidelobes of a pattern alternate sign compared to the main beam. These sidelobes which are phased at 180^o will cancel the main beam of the cosine square pattern and give a narrower main beam for the composite. The second sidelobe of the uniform distribution occurs in the range of the first sidelobe of the cosine squared pattern. They will cancel because they are 180^o out of phase. The third advantage to the pedestal is that the field does not go to zero at the edge of the aperture. This can become difficult to achieve. If the aperture is approximated by an array, the power division network must supply a large ratio across the aperture.

We find the Fourier transform by using the sum rule. The Fourier transform of a sum is the sum of the Fourier transforms. The k space pattern is the sum of the two k space transforms for the uniform and cosine squared aperture distributions. The distribution must be normalized when this is done.

On page 542 is a k space pattern of a distribution with a pedestal 10 dB below the peak. Comparing this with the pattern on page 541, we see that the main beam has decreased but the sidelobes have increased. These increased sidelobes come from the uniform distribution. There is an extra sidelobe next to the main beam. If we decrease the size of the pedestal, then we can expect the outer sidelobes to decrease but the main beam to


Aperture Distribution
Cos(π x/a)
Level dB
-10
-20
-30
-40
-50
-60
πa/λ (Sin θ - Sin θo)
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.
Aperture Distribution:
Cos²(πx/a)
a/2
-a/2
θ
Level dB
-10
-20
-30
-40
-50
-60
πa/λ (Sin θ - Sin θo)


0.684 Cos² (π x/a) + 0.316
10 db Pedestal
Level dB
πa/λ (Sin θ - Sin θo)
-10
-20
-30
-40
-50
-60
-5 -4 -3 -2 -1 0 1 2 3 4 5
-25 -20 -15 -10 -5 0 5 10 15 20 25
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

increase. A k space pattern for a pedestal height of 15 dB below the aperture distribution peak is plotted on page 544. The sidelobes have decreased to 33.4 dB which is lower than the cosine squared distribution and the beamwidth is still lower. Finually a 20 dB pedestal reduces the maximum sidelobe level to 40.2 dB below the main beam as shown on page 545. The beamwidth is still slightly less than the cosine squared distribution. The maximum sidelobe will not decrease indefinitely but reaches a minimum for a pedestal level at about 22 dB which can be seen from the plot on page 546 of the maximum sidelobe versus the pedestal level. The beamwidth factor is also plotted on the same graph. It is less than the beamwidth factor for the cosine squared aperture distribution (1.44) over the range of pedestal levels on the plot.

The aperture distribution is given by

$$E(x) = PD + (1 - PD)\,\mathrm{Cos}^2(\pi x/a) \qquad |x| \leq a/2$$

where PD is the voltage pedestal level.

This distribution is a handy approximation for tolerance studies on the more difficult to obtain Dolph-Tchebyscheff array distribution where the array values are samples of the continuous distribution.

Taylor Distribution

The Taylor distribution seeks to lower the inner sidelobes of the uniform distribution much in the same way the cosine squared distribution added to the pedestal reduces the sidelobes of the pedestal. Taylor achieved this by modifying some of the zeros of the k space pattern of the distribution to lower the sidelobes. The aperture distribution is expanded in a Fourier cosine series and a match is made between the assumed aperture field and the k space pattern by a Fourier series expansion in k space.

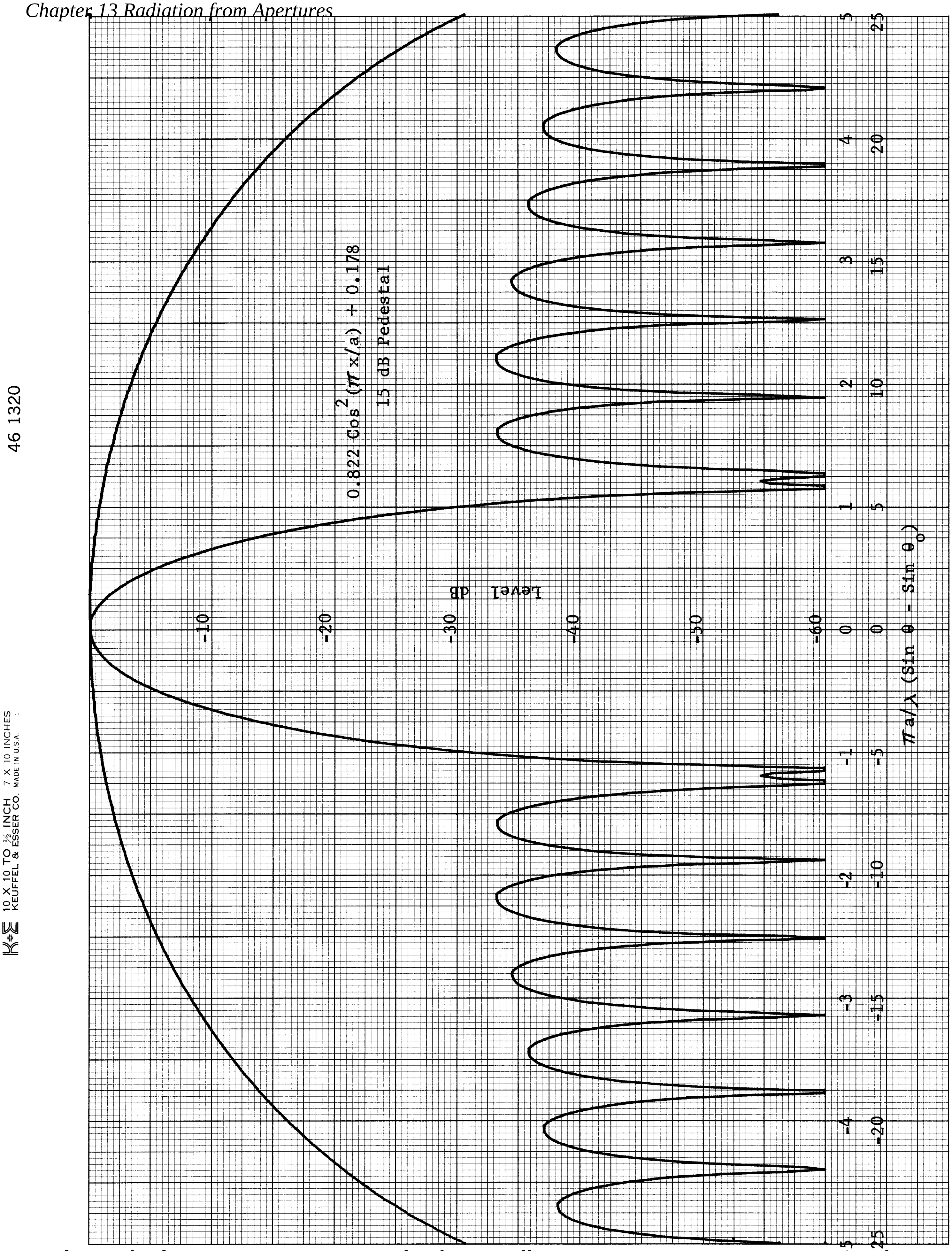
The uniform aperture distribution has the following k space pattern:

$$\frac{\mathrm{Sin}\,\pi U}{\pi U}$$

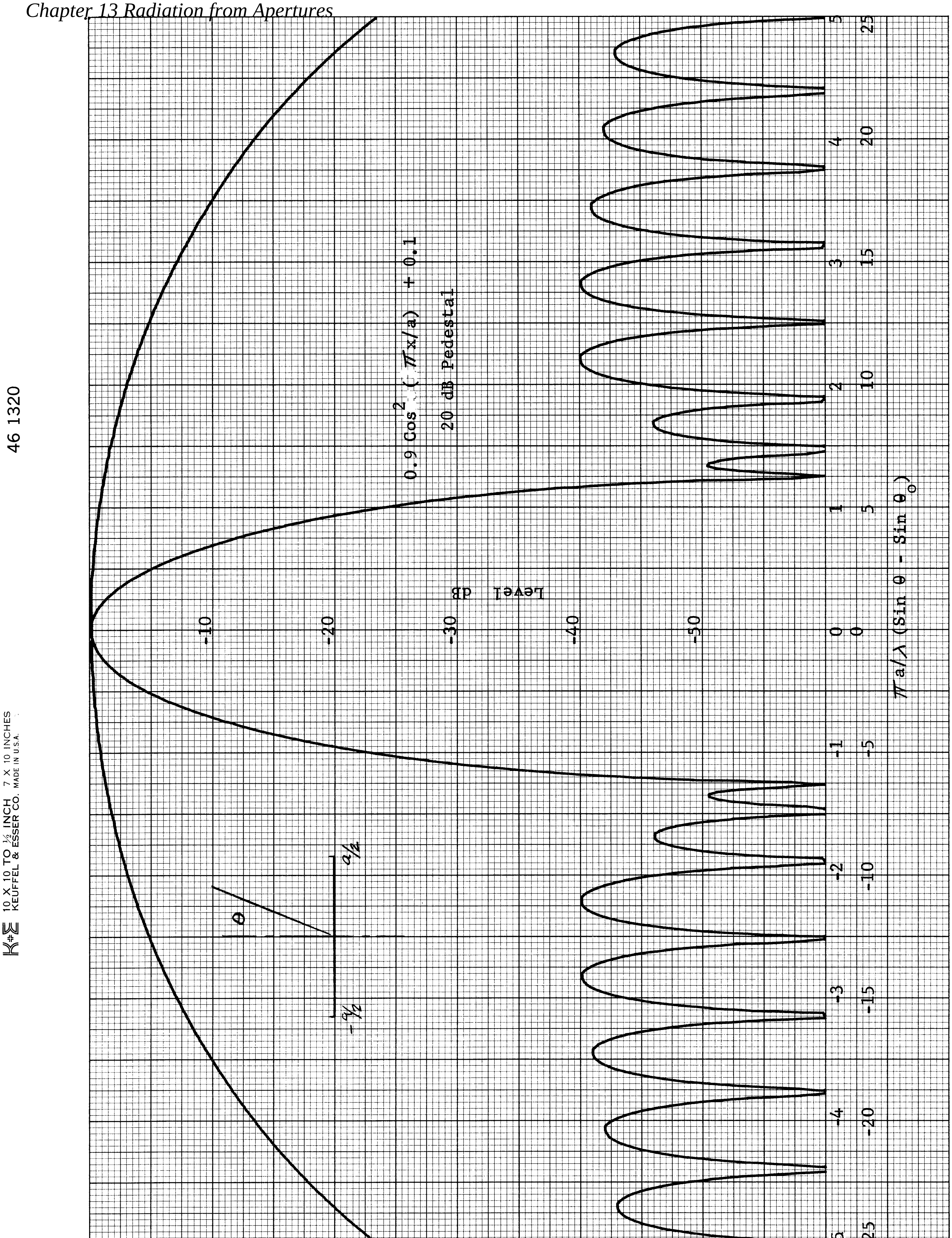
where the substitution $U = a/\lambda\,(\mathrm{Sin}\,\theta - \mathrm{Sin}\,\theta_o)$ has been made. The nulls of the pattern are located at integer values of U. A number of the close in nulls are picked to be modified to lower the near by sidelobes. Pick an integer, $\bar{n}$, such that the nulls for $|U| \geq \bar{n}$ are located at integer values as before. We remove the $\bar{n} - 1$ nulls from the uniform distribution by dividing by the product of factors: $(1 - U^2/N^2)$, $N = 1, \ldots, \bar{n} - 1$.

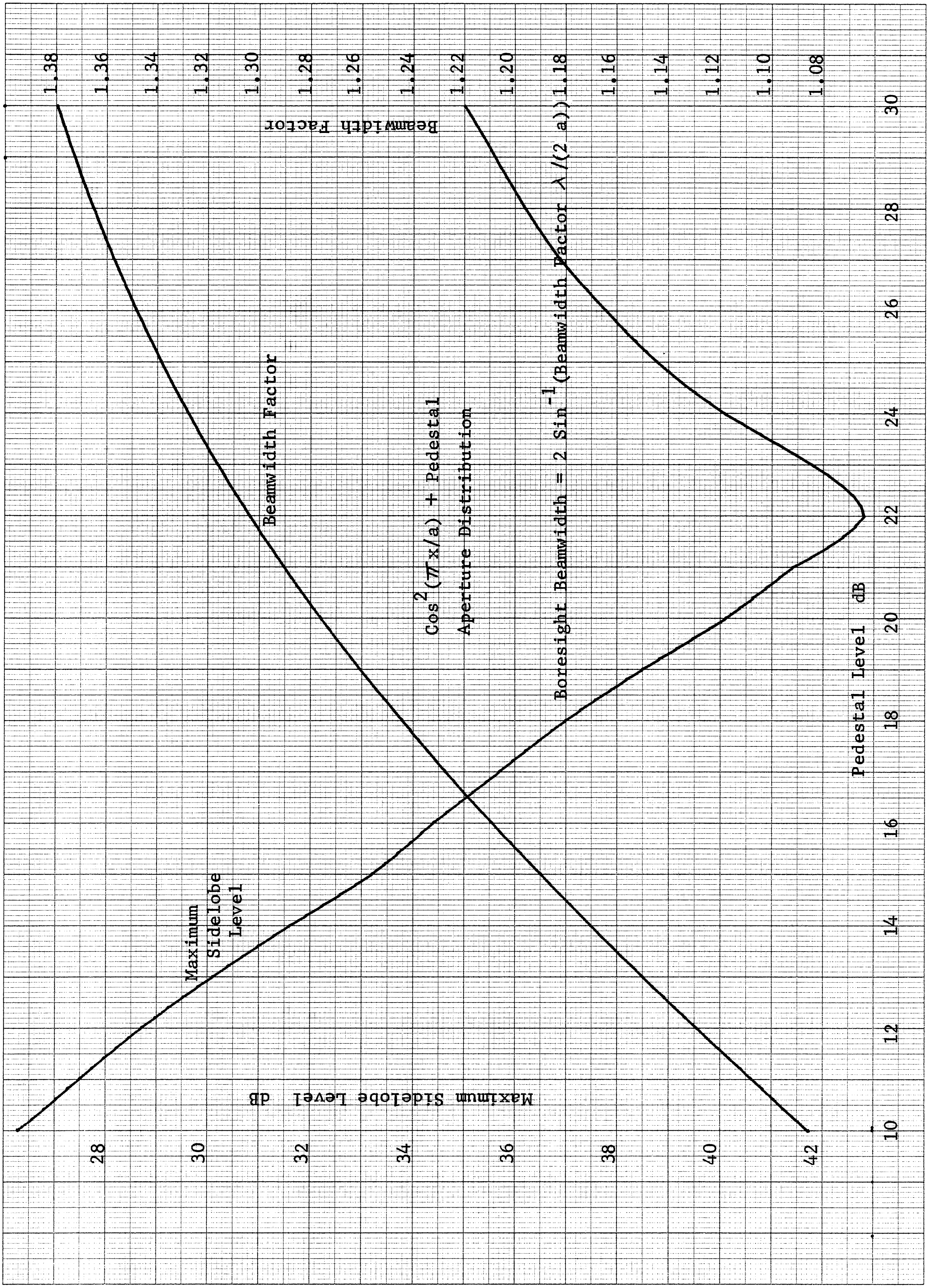
$$\frac{\mathrm{Sin}\,\pi U}{\pi U \prod_{N=1}^{\bar{n}-1} (1 - U^2/N^2)}$$

We then add back modified nulls, U_n, to get the final k space pattern.



0.822 Cos² (π x/a) + 0.178
15 dB Pedestal
Level dB
πa/λ (Sin θ - Sin θo)
-10
-20
-30
-40
-50
-60
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


0.9 Cos2 (πx/a) + 0.1
20 dB Pedestal
Level dB
-10
-20
-30
-40
-50
πa/λ (Sin θ - Sin θo)
-5 -4 -3 -2 -1 0 1 2 3 4 5
-25 -20 -15 -10 -5 0 5 10 15 20 25
a/2
-a/2
θ
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


Beamwidth Factor
1.38 1.36 1.34 1.32 1.30 1.28 1.26 1.24 1.22 1.20 1.18 1.16 1.14 1.12 1.10 1.08
Beamwidth Factor
Cos^2(πx/a) + Pedestal
Aperture Distribution
Boresight Beamwidth = 2 Sin^{-1}(Beamwidth Factor λ/(2 a))
Pedestal Level dB
10 12 14 16 18 20 22 24 26 28 30
Maximum
Sidelobe
Level
Maximum Sidelobe Level dB
28 30 32 34 36 38 40 42

$$f(U) = \frac{\sin \pi U}{\pi U} \frac{\prod_{n=1}^{\bar{n}-1} (1 - U^2/U_n^2)}{\prod_{N=1}^{\bar{n}-1} (1 - U^2/N^2)}$$

Taylor determined the location of the new nulls from the Tchebyscheff polynomials in an approximation to the Dolph Tchebyscheff array and gave the following formula for the nulls.

$$U_n = \bar{n}\left(\frac{A^2 + (N - \frac{1}{2})^2}{A^2 + (\bar{n} - \frac{1}{2})^2} \right)^{\frac{1}{2}} \qquad N = 1, \ldots, \bar{n} - 1$$

A is found from the expression: $\cosh \pi A = b$, with $20 \log_{10} b$ = Sidelobe Level.

This gives us the pattern in k space, but we must still find the aperture distribution that will give this pattern. The aperture distribution may be found by expanding it in a Fourier cosine series.

$$E(x) = \sum_{m=0}^{\infty} B_m \cos(2m\pi x/a)$$

The pattern is found from the Fourier transform.

$$f(k_x) = \int_{-a/2}^{a/2} E(x)\, e^{jk_x x}\, dx$$

$$f(u) = \int_{-a/2}^{a/2} E(x)\, e^{j2\pi u x/a}\, dx$$

Substituting the expression for $E(x)$ and reversing the order of the sum and integral we get

$$f(u) = \sum_{m=0}^{\infty} B_m \int_{-a/2}^{a/2} \cos(2m\pi x/a)\, e^{j2\pi u x/a}\, dx$$

Since the aperture distribution is an even function, the odd part of the integral will be zero and the expression for the k space pattern becomes

$$f(u) = \sum_{m=0}^{\infty} B_m \int_{-a/2}^{a/2} \cos(2m\pi x/a) \cos(2\pi u x/a)\, dx$$

We can find the coefficients, B_m, by matching the patterns at integer values of U. The integral is zero unless $U = m$

$$a B_o = f(0)$$

$$a/2\ B_m = f(m) \qquad m = 1, 2, \ldots$$

But we have only modified the location of the first $\bar{n} - 1$ zeros of the U space pattern.

$$f(m) = 0 \quad \text{for} \quad m \geq \bar{n}$$

The Fourier cosine series of the aperture distribution has only $\bar{n}$ components.

$$E(x) = \left(f(0) + 2 \sum_{m=1}^{\bar{n}-1} f(m)\cos(2m \quad x/a)\right)/a$$

When we substitute integer values, m, in the expression for f(U), we are left with indeterminate expressions such as

$$\frac{\sin(0)}{0} = f(0) = 1$$

We can resolve these by using L'Hospital's rule.

$$f(U) = g(U)/h(U)$$

$$g'(u) = \pi \cos \pi u \prod_{N=1}^{\bar{n}-1} (1 - u^2/u_N^2) - 2u \sin \pi u \sum_{k=1}^{\bar{n}-1} \frac{1}{u_k^2} \prod_{\substack{N=1 \\ N \neq k}}^{\bar{n}-1} (1 - u^2/u_N^2)$$

$$h'(u) = \pi \prod_{N=1}^{\bar{n}-1} (1 - u^2/N^2) - 2u^2\pi \sum_{k=1}^{\bar{n}-1} \frac{1}{k^2} \prod_{\substack{N=1 \\ N \neq k}}^{\bar{n}-1} (1 - u^2/N^2)$$

$$f(m) = g'(m)/h'(m)$$

$$g'(m) = \pi \cos \pi m \prod_{N=1}^{\bar{n}-1} (1 - m^2/u_N^2) = (-1)^m \pi \prod_{N=1}^{\bar{n}-1} (1 - m^2/u_N^2)$$

Since $\sin \pi m = 0$

In h'(m) the term $\pi \prod_{N=1}^{\bar{n}-1} (1 - m^2/N^2) = 0$ since m = N for some N. In the second term of h'(m) if $k \neq m$, then the term is zero from the product of factors $(1 - m^2/N^2)$. The factor h'(m) becomes

$$\frac{-2m^2\pi}{m^2} \prod_{\substack{N=1 \\ N \neq m}}^{\bar{n}-1} (1 - m^2/N^2)$$

The expression for the coefficients becomes

$$f(m) = \frac{(-1)^m \prod_{N=1}^{\bar{n}-1} (1 - m^2/u_N^2)}{-2 \prod_{\substack{N=1 \\ N \neq m}}^{\bar{n}-1} (1 - m^2/N^2)}$$

With these expressions we can find the aperture distribution to give the desired sidelobe level.

A Taylor aperture distribution was designed for a 30 dB maximum sidelobe level by modifying 5 zeros of the uniform distribution or $\bar{n} = 6$. $\bar{n}$ is the first non-modified zero of the uniform distribution. The result of the calculations above is the aperture distribution given on page 550. The k space pattern is plotted on page 551 for this aperture distribution and shows a peak sidelobe level of 30 dB. A second design was performed with $\bar{n} = 12$, that is, there are 11 modified zeros of the uniform distribution. The k space pattern of this design is plotted on page 552. By comparing the two patterns we can see that the sidelobes do not tail off as fast for $\bar{n} = 12$ as for the design with $\bar{n} = 6$. This is because the Dolph Tchebysheff array is more closely approximated with $\bar{n} = 12$. The array has equal sidelobes everywhere. The second point is that the beamwidth for $\bar{n} = 12$ is less than the design with $\bar{n} = 6$. The array has the minimum beamwidth for a given sidelobe level.

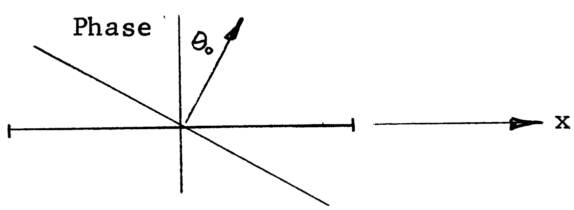
$$\text{Beamwidth (k Space)} = 1.1166\ a/\lambda \qquad \bar{n} = 6$$

$$\text{Beamwidth (k Space)} = 1.0937\ a/\lambda \qquad \bar{n} = 12$$

Larger number of modified zeros will give smaller beamwidths as the design approaches a Dolph Tchebyscheff array distribution which has equal sidelobes. We can use this aperture distribution to design arrays which are sampled apertures. This has an advantage over the array because it has a limiting directivity for a given sidelobe level. The limit in a large Dolph Tchebysheff array to the directivity is 1/(Sidelobe Level). The sampled Taylor distribution does not have this restriction.

LINEAR PHASE TAPER

Suppose there is a linear phase taper across the aperture as shown in the figure. We would expect the pattern maximum to be in the direction θ_o for



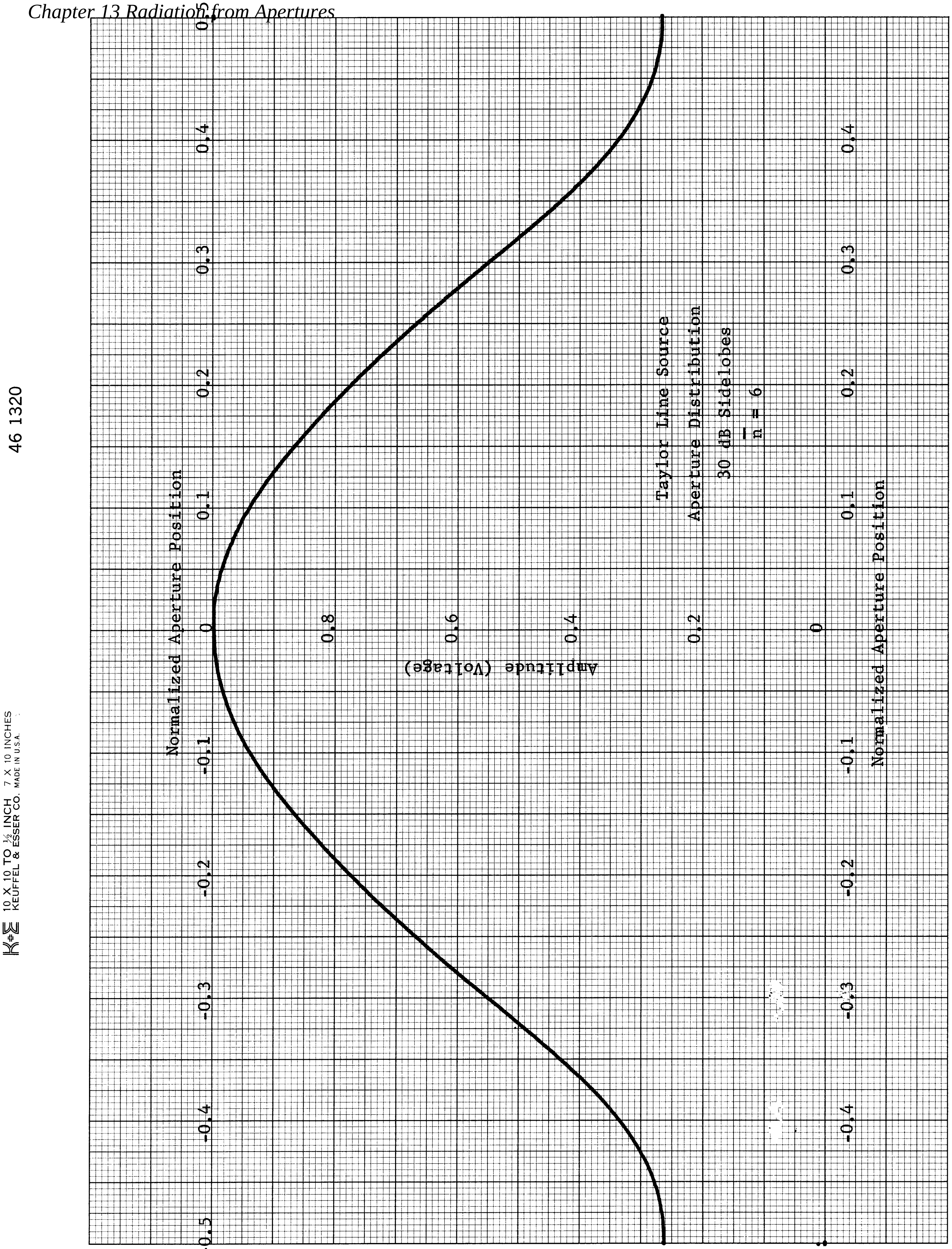
a phase taper: $-\ k\ x\ \mathrm{Sin}\ \theta_o$. Substitute this aperture excitation into the Fourier integral to obtain the k space pattern.

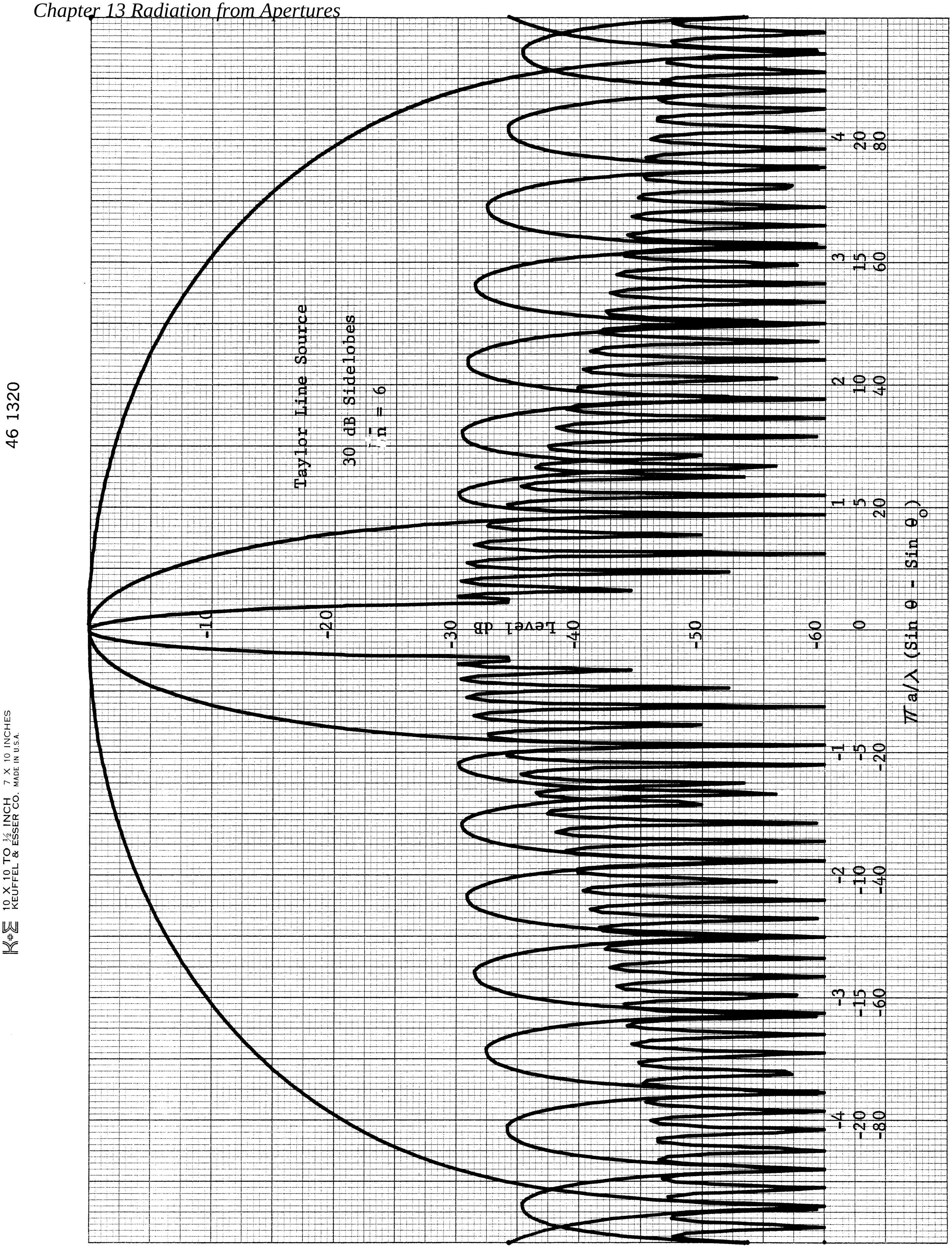
$$f(k_x) = \int_{-a/2}^{a/2} E_o(x)\, e^{jkx(\sin\theta - \sin\theta_o)}\, dx$$

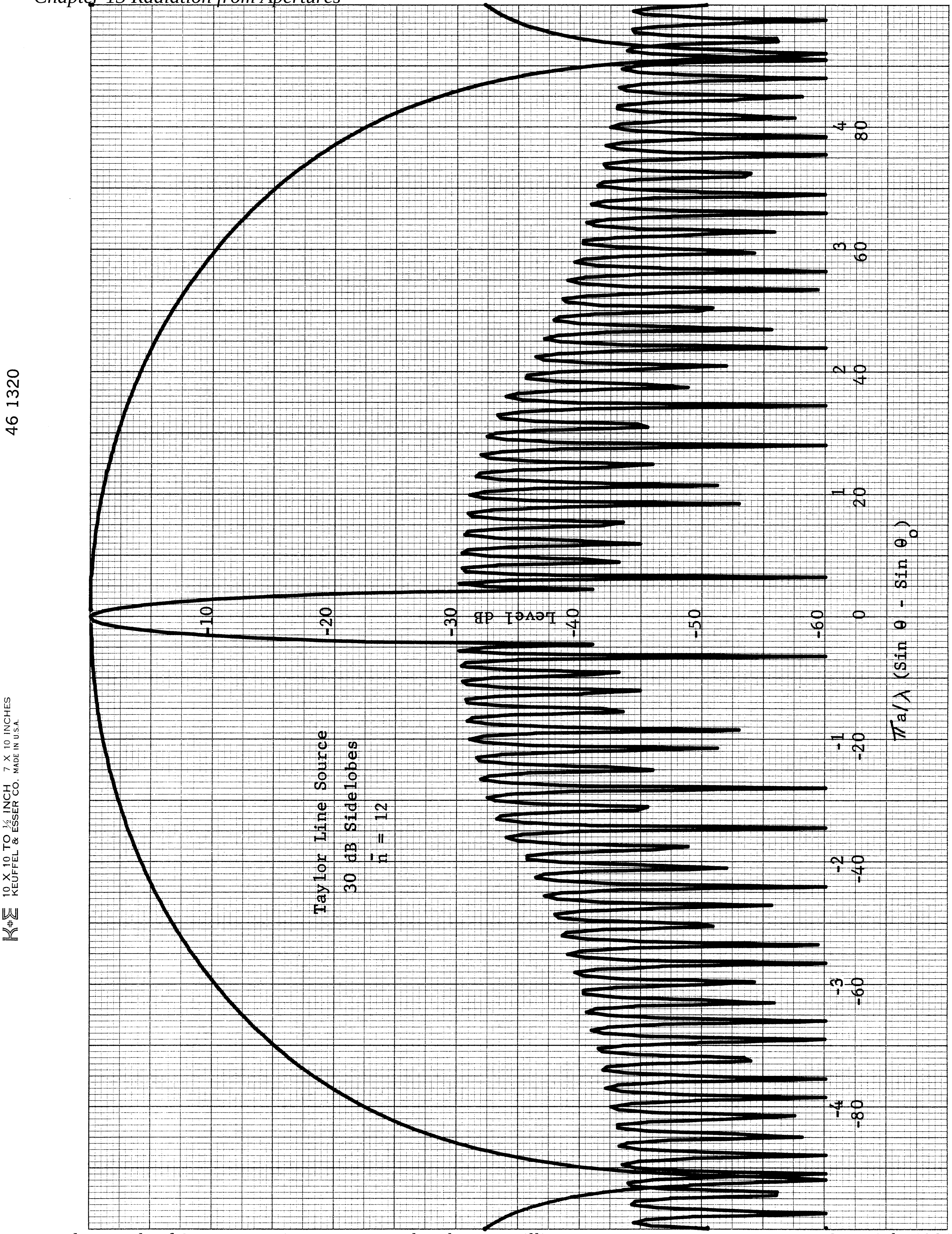
For a uniform amplitude distribution the result of the integral is

$$f(k_x) = 2\ E_o a \frac{\mathrm{Sin}\left(\frac{ka}{2}(\mathrm{Sin}\ \theta - \mathrm{Sin}\ \theta_o)\right)}{\frac{ka}{2}(\mathrm{Sin}\ \theta - \mathrm{Sin}\ \theta_o)}$$

This is equivalent to shifting the origin in k_x space as shown on all the k space patterns. Shifting the origin in k space broadens the beam in real space. The


Taylor Line Source
Aperture Distribution
30 dB Sidelobes
$\bar{n} = 6$
Normalized Aperture Position
Amplitude (Voltage)
Normalized Aperture Position


Taylor Line Source
30 dB Sidelobes
$\bar{n}$ = 6
Level dB
-10
-20
-30
-40
-50
-60
0
1
5
20
2
10
40
3
15
60
4
20
80
-1
-5
-20
-2
-10
-40
-3
-15
-60
-4
-20
-80
π a/λ (Sin θ - Sin θ_o)
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


Taylor Line Source
30 dB Sidelobes
$\bar{n}$ = 12
Level dB
-10
-20
-30
-40
-50
-60
$\pi a/\lambda$ (Sin θ - Sin θ_o)
-4 -80
-3 -60
-2 -40
-1 -20
0
1 20
2 40
3 60
4 80
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

broadening is approximately due to the Cos θ_o projection of the aperture in the direction of the main beam. A plot of the ratio of the beamwidth at boresight to the beamwidth for a scanned beam is given on page 554. Here the center of the beam is taken as the mean between the two beam edges and not necessarily at the peak of the beam. As the beam approaches endfire, a point is reached where the beamwidth defining level reaches 90^o and we must say that the beam is endfire even though the peak is not at 90^o. We can say that there is another beam below the aperture which is also approaching 90^o. The two beams merge into a single endfire beam. This is the limit of scan which increases with increasing size of the aperture. A plot of the 3 dB beamwidth versus the aperture size for various scan angles is given on page 555. The aperture distribution is uniform. The curve is the same for other aperture distributions only it is moved up. On this curve the scan limit shows. Apertures less than about 15 wavelengths cannot be scanned to 80^o because the beam becomes endfire before the angle is reached.

Suppose the 3 dB beamwidth point is given by an expression

$$\frac{k\ a}{2} \operatorname{Sin} \theta = \pm C$$

for a symmetrical distribution in the aperture. The scanned beam is given by

$$\frac{K\ a}{2} (\operatorname{Sin} \theta - \operatorname{Sin} \theta_o) = \pm C$$

$$\operatorname{Sin} \theta = \operatorname{Sin} \theta_o \pm \frac{2\ C}{k\ a} \quad \text{when} \quad \operatorname{Sin} \theta + \frac{2\ C}{k\ a} < 1$$

When the inequality is equal to one, the 3 dB (or whatever level specified by C) level is at endfire and the beamwidth will have a discontinuous jump. The edges of the pattern beamwidth are found from

$$\operatorname{Sin} \theta_1 = \operatorname{Sin} \theta_o - \frac{2\ C}{k\ a}$$

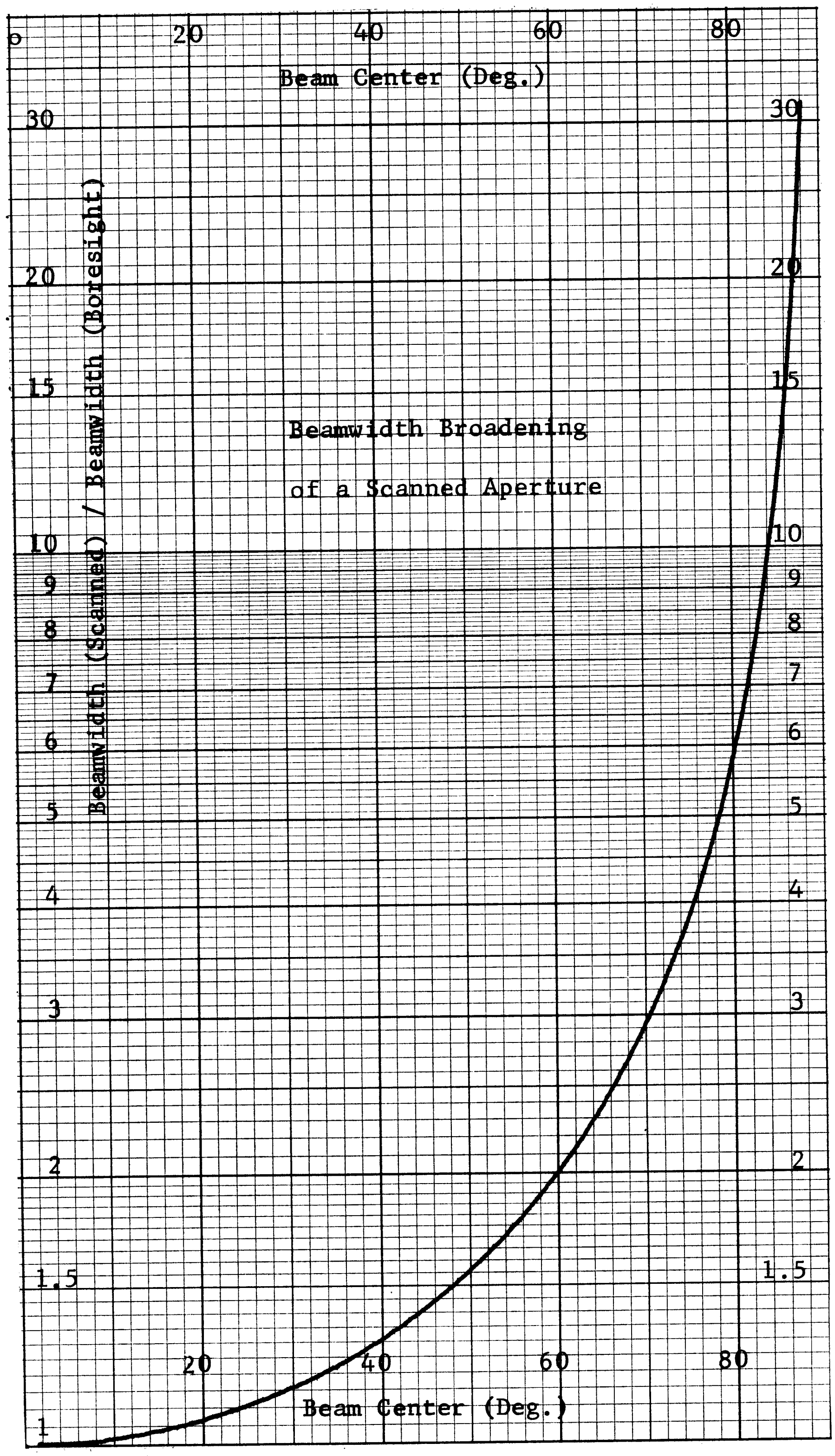
$$\operatorname{Sin} \theta_2 = \operatorname{Sin} \theta_o + \frac{2\ C}{k\ a}$$

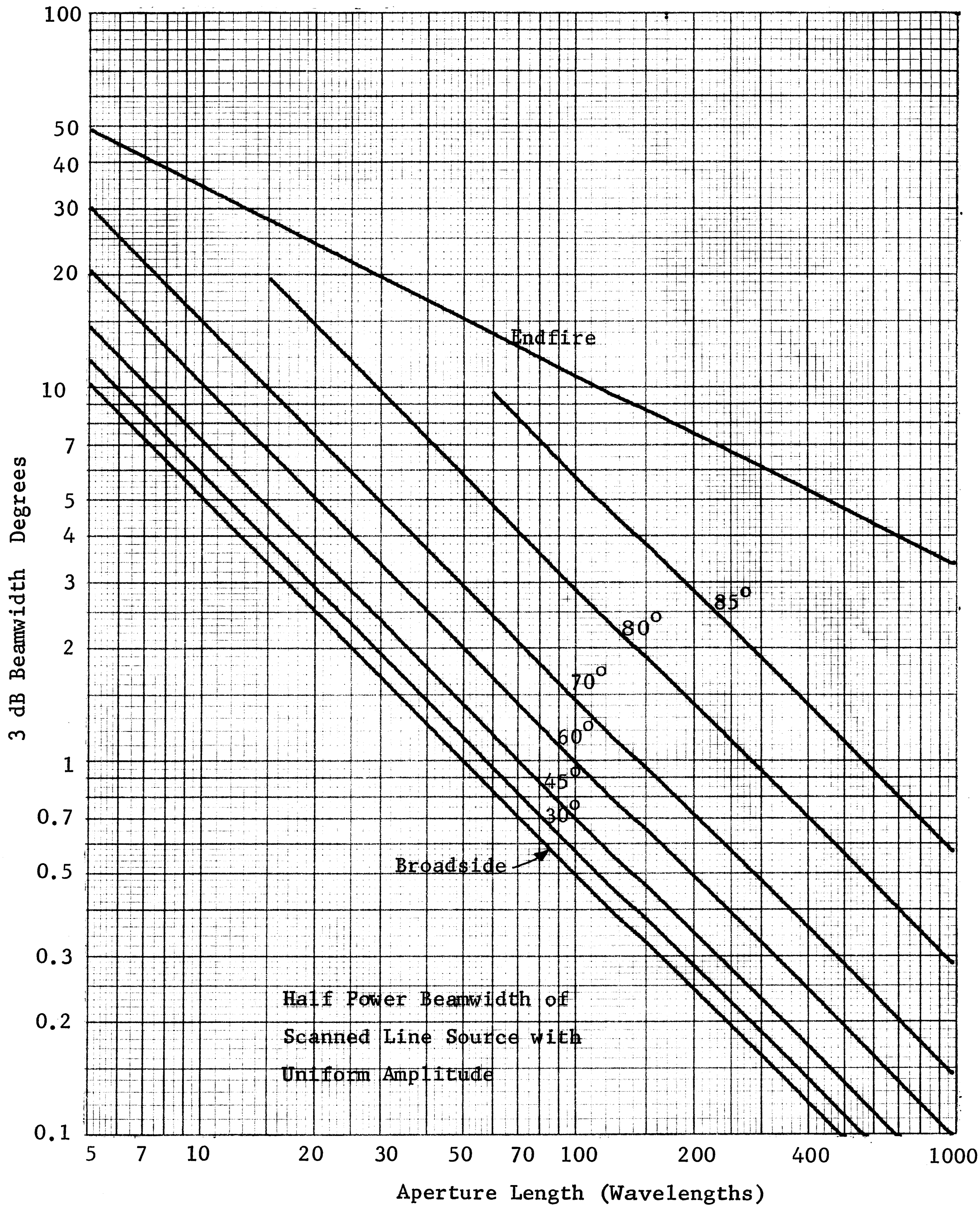
If $\operatorname{Sin} \theta_o + 2\ C/(k\ a) \geq 1$, then the beamwidth is $2(\theta_1 - \pi/2)$ provided $\operatorname{Sin} \theta_o \leq 1$.

It is possible for the beam to be scanned into invisible space and Sin θ_o will be greater than 1. This means that the phase taper along the line source is greater than necessary to produce an endfire pattern such as the Woodyard and Hansen endfire array (aperture). The peak of the beam no longer occurs at $\operatorname{Sin} \theta - \operatorname{Sin} \theta_o = 0$ since Sin θ is limited to one. The peak of the beam in k space occurs at

$$k_x a/2 = \operatorname{Sin} \theta_o - 1$$

The effect is to raise the sidelobes since the peak no longer occurs at zero dB on the k space patterns and narrow the beamwidth. Remember we said that the portion of the k space pattern in the invisible region represents stored energy in the near field. The stored energy will raise the Q of the antenna which reduces the frequency bandwidth and decreases the efficiency due to material losses.


Beam Center (Deg.)
0
20
40
60
80
Beamwidth (Scanned) / Beamwidth (Boresight)
30
20
15
10
9
8
7
6
5
4
3
2
1.5
1
Beamwidth Broadening
of a Scanned Aperture
20
40
60
80
Beam Center (Deg.)


100
50
40
30
20
10
7
5
4
3
2
1
0.7
0.5
0.3
0.2
0.1
3 dB Beamwidth Degrees
5
7
10
20
30
50
70
100
200
400
1000
Aperture Length (Wavelengths)
Endfire
85°
80°
70°
60°
45°
30°
Broadside
Half Power Beamwidth of
Scanned Line Source with
Uniform Amplitude

CIRCULAR APERTURES

Many common apertures conform to circles. The same two dimensional Fourier transform relation holds but many of these can be separated in r and ϕ', and the integral can be separated into single integrals. In polar coordinates the integral becomes

$$f(\theta,\phi) = \int_0^{2\pi}\int_0^{a} E(\rho,\phi')\, e^{jk\rho \sin\theta \cos(\phi-\phi')} \rho \, d\rho \, d\phi'$$

where a is the radius of the aperture. This integral leads to a k_ρ space. Suppose we have a circularly symmetrical distribution. Because there is no ϕ' dependence in the distribution, the integral can be found exactly.

$$f(k_\rho) = \int_0^{2\pi}\int_0^{a} E(\rho)\, e^{jk\rho \sin\theta \cos(\phi-\phi')} \rho \, d\rho \, d\phi'$$

$$\int_0^{2\pi} e^{jk\rho \sin\theta \cos(\phi-\phi')} d\phi' = 2\pi J_0(k\rho \sin\theta)$$

$J_0(x)$ is the zeroth order Bessel function of the first kind. The Fourier integral for the circularly symmetric distribution becomes

$$f(k_\rho) = 2\pi \int_0^{a} E(\rho) J_0(k\rho \sin\theta) \rho \, d\rho$$

and all ϕ patterns are identical.

For a uniform aperture field this integral gives the k space pattern.

$$f(k_\rho) = 2\pi a^2 \frac{J_1(ka \sin\theta)}{ka \sin\theta}$$

The k space pattern is plotted on page 557.

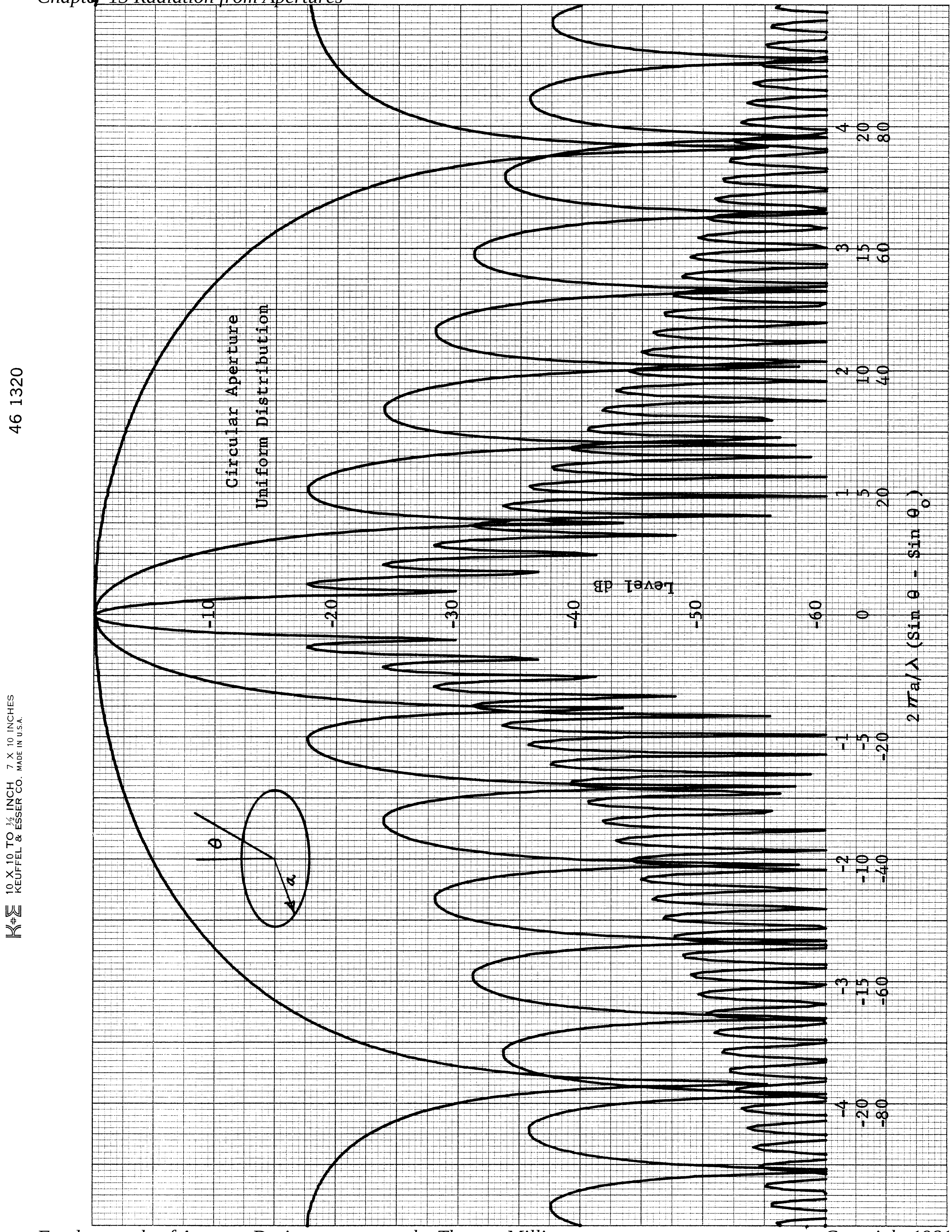
Beamwidth (k Space) = $1.029 \; 2a/\lambda$

Maximum Sidelobe = 17.6 dB

The most common tapered circularly symmetrical distribution is a parabolic to the n-th power.

$$E(\rho) = (1 - (\rho/a)^2)^n$$

We consider this because it has a simple Fourier transform and can approximate tapered circular aperture distributions reasonably well.


Circular Aperture
Uniform Distribution
Level dB
-10
-20
-30
-40
-50
-60
2πa/λ (Sin θ - Sin θo)
θ
a
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

$$f(k_\rho) = \frac{2^{n+1}(n+1)!\ J_{n+1}(k\ a\ \sin\theta)}{(k\ a\ \sin\theta)^{n+1}} \quad \text{Normalized Pattern}$$

The first three orders of patterns are plotted on pages 559 - 561 in k space. The scale used in these patterns is similiar to the scale used for the rectangular distributions so that comparisons can be easily made. From these plots we find the maximum sidelobe level and the beamwidth factor.

n	Beamwidth Factor	Sidelobe Level
0	1.029	17.6 dB
1	1.270	24.7
2	1.473	30.6
3	1.651	36.0

The beamwidth is found from the following formula for $n = 0$ (uniform)

$$\text{Beamwidth} = 2\ \sin^{-1}(1.029\ \lambda/4\ a)$$

with similiar formulas for the other cases of n.

PARABOLIC^N DISTRIBUTION ON A PEDESTAL

When a cosine squared distribution was placed on a pedestal for a rectangular aperture, the maximum sidelobe was reduced. The second sidelobe of the uniform distribution is located at about the same place as the first sidelobe of the cosine squared distribution in k space and since the signs of the sidelobes alternate, they cancelled. We can do the same thing by adding a pedestal to the parabolic to the n-th power distribution for circular apertures. We find the pattern by adding the Fourier transforms. The aperture distribution is

$$E(\rho) = PD + (1 - PD)(1 - (\rho/a)^2)^n$$

The unnormalized Fourier transform of the parabolic distribution is given by

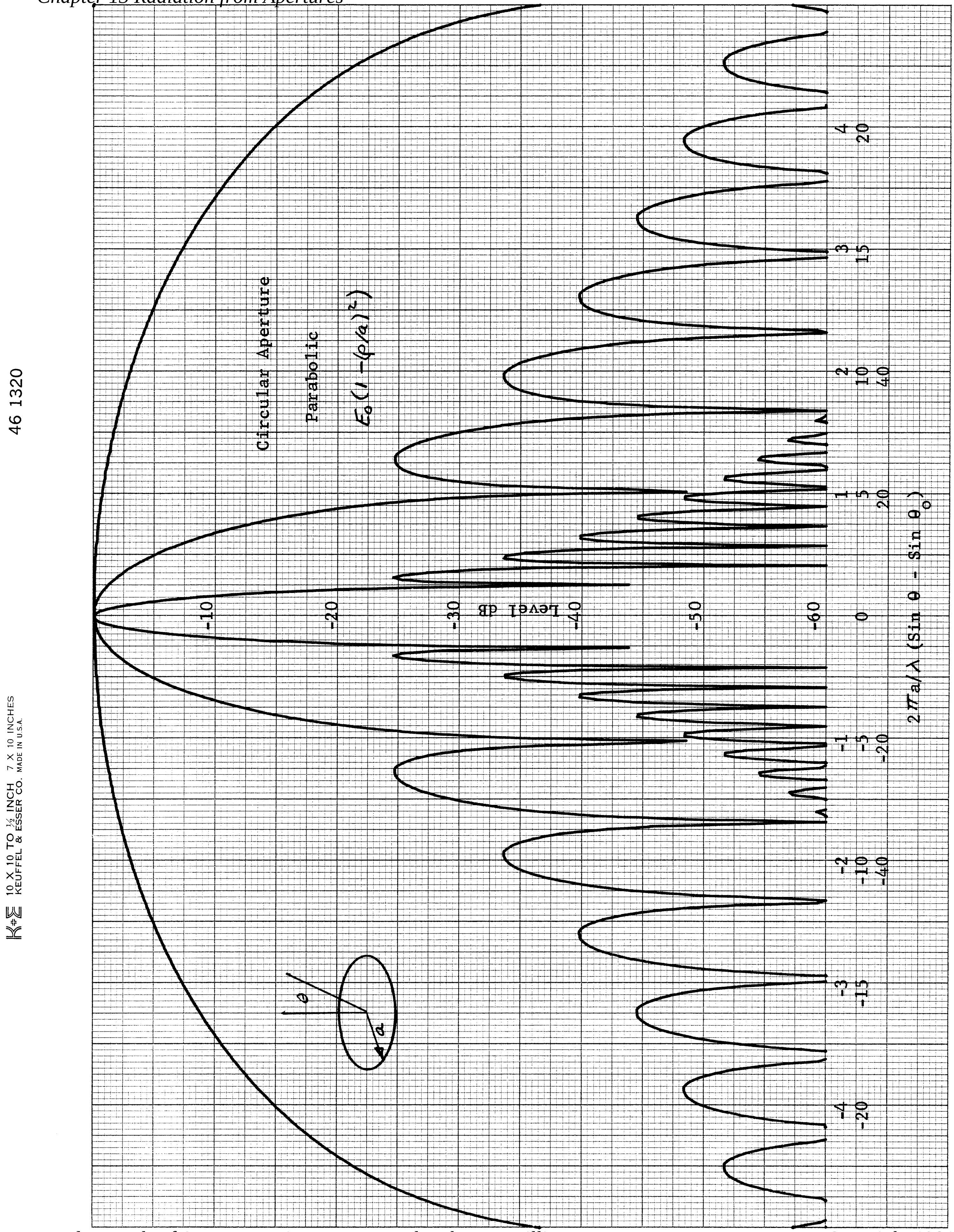
$$f(k_\rho) = \frac{2^{n+1} n!\ J_{n+1}(k_\rho\ a)}{(k_\rho\ a)^{n+1}}$$

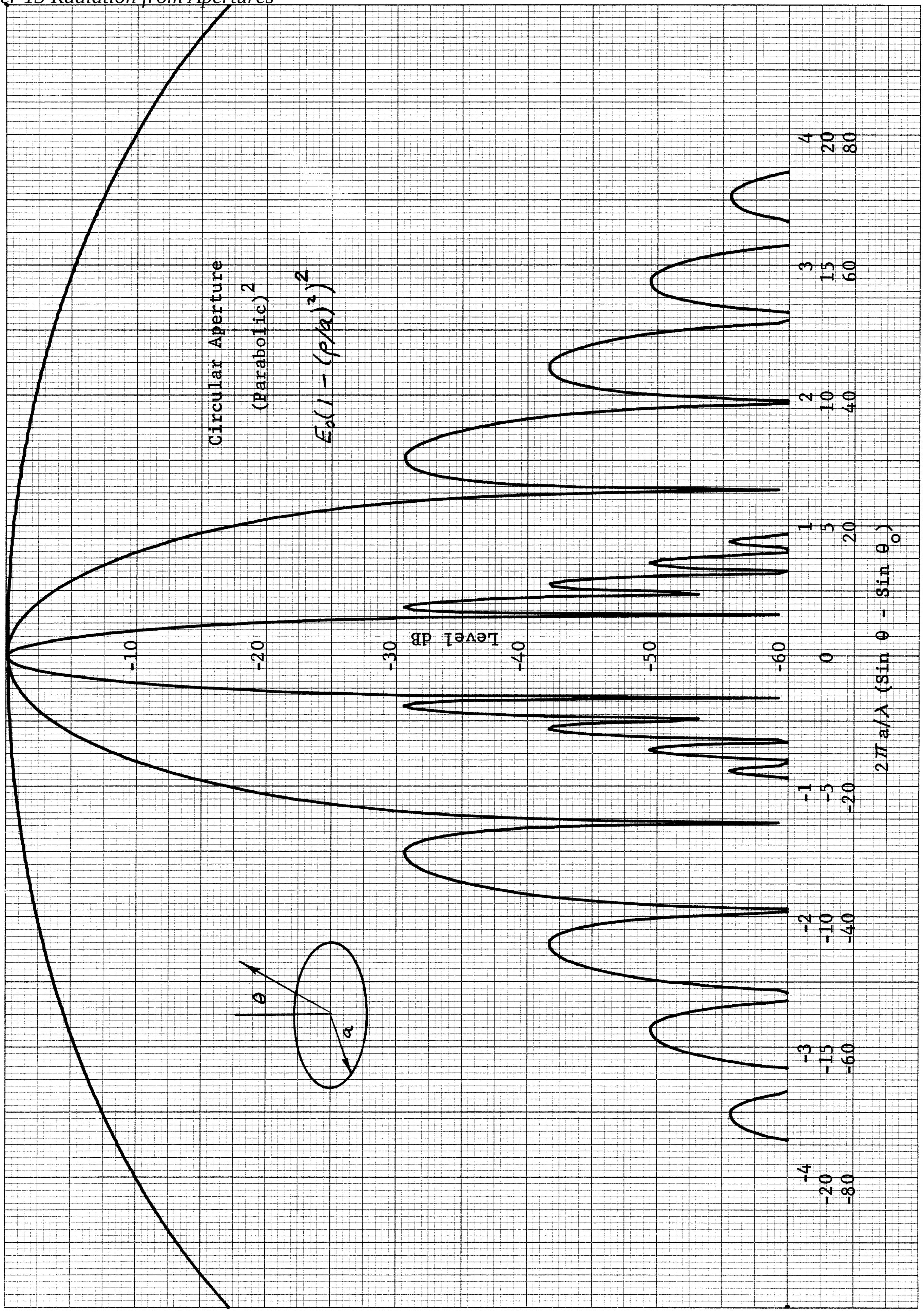
which includes $n = 0$ (the uniform distribution). The unnormalized Fourier transform of the sum is

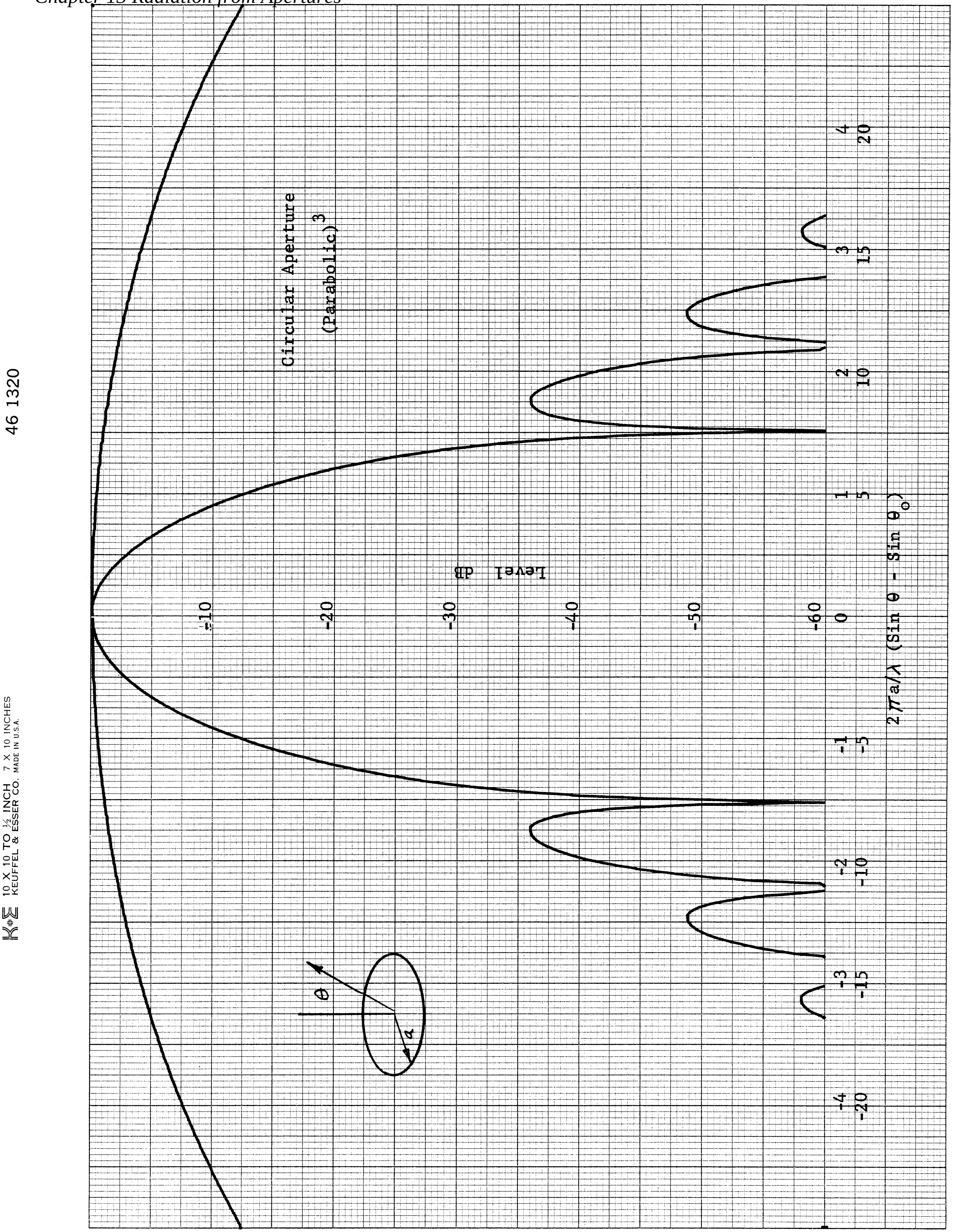
$$PD\ \frac{2\ J_1(k_\rho\ a)}{k_\rho\ a} + (1 - PD)\ \frac{2^{n+1} n!\ J_{n+1}(k_\rho\ a)}{(k_\rho\ a)^{n+1}}$$

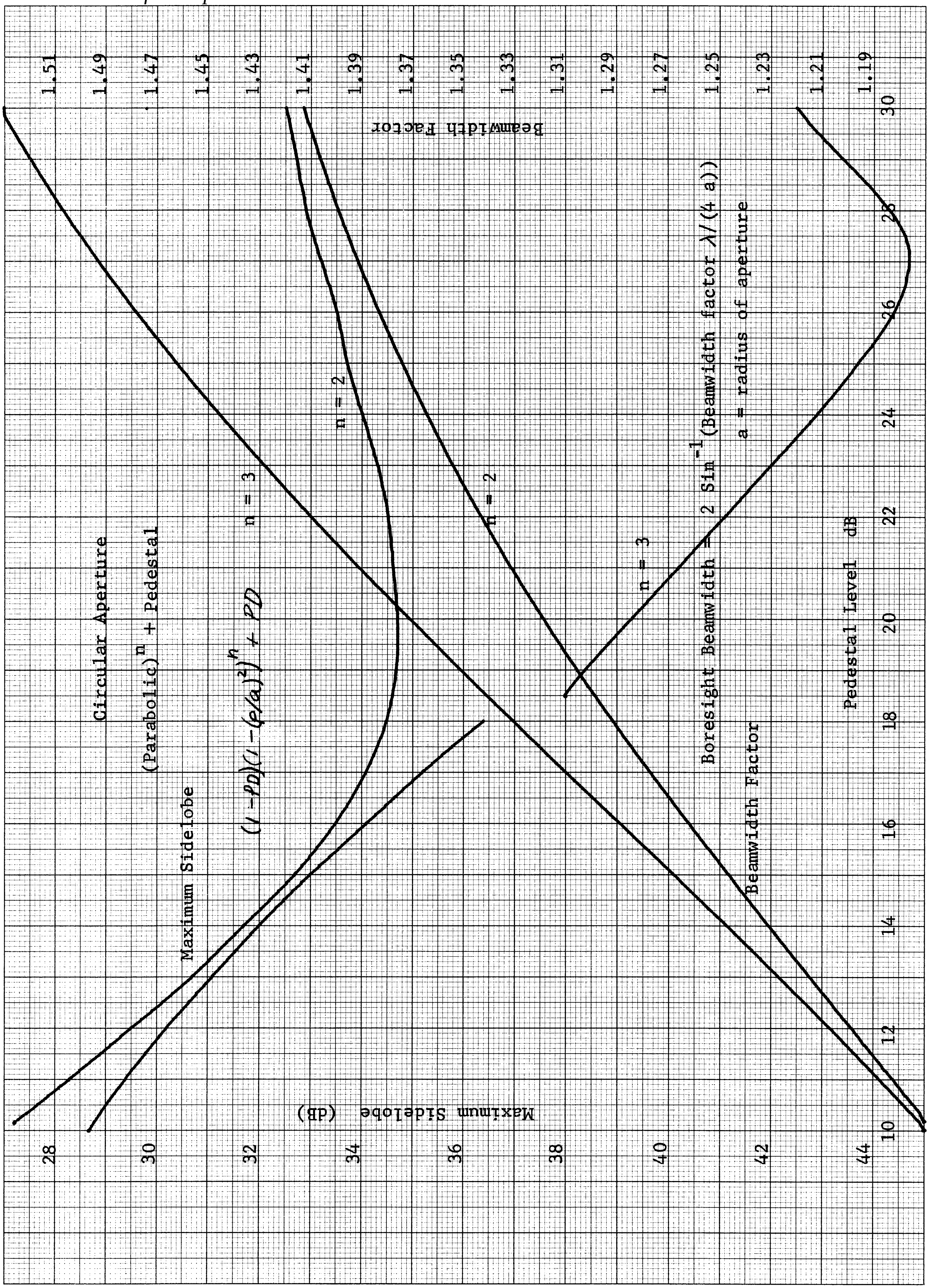
At $k_\rho a = 0$, this becomes: $PD + (1 - PD)/(n + 1)$ and is the normalization factor.

A series of patterns were drawn for $n = 2$ and $n = 3$ from which the maximum sidelobe and beamwidth factor were found for various pedestal heights. The results of these patterns are plotted on page 562. This plot shows that the minimum sidelobes occur for $n = 3$. If we compare the plot of the k space


Circular Aperture
Parabolic
$E_o(1-(\rho/a)^2)$
Level dB
-10
-20
-30
-40
-50
-60
$2\pi a/\lambda$ (Sin θ - Sin θ$_o$)
46 1320
10 X 10 TO 1/2 INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


Circular Aperture
(Parabolic)²
E₀(1 − (ρ/a)²)²
Level dB
-10
-20
-30
-40
-50
-60
0
1 5 20
2 10 40
3 15 60
4 20 80
-1 -5 -20
-2 -10 -40
-3 -15 -60
-4 -20 -80
2π a/λ (Sin θ − Sin θo)
θ
a


Circular Aperture
(Parabolic)^3
Level dB
$2\pi a/\lambda$ (Sin θ - Sin θ$_o$)
-10
-20
-30
-40
-50
-60
-4 -20
-3 -15
-2 -10
-1 -5
0
1 5
2 10
3 15
4 20
θ
a
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


Circular Aperture
(Parabolic)^n + Pedestal
Maximum Sidelobe
(1 - PD)(1 - (ρ/a)^2)^n + PD
n = 3
n = 2
n = 2
n = 3
Beamwidth Factor
Boresight Beamwidth = 2 Sin^-1 (Beamwidth factor λ/(4 a))
a = radius of aperture
Beamwidth Factor
Pedestal Level dB
Maximum Sidelobe (dB)
1.51
1.49
1.47
1.45
1.43
1.41
1.39
1.37
1.35
1.33
1.31
1.29
1.27
1.25
1.23
1.21
1.19
10
12
14
16
18
20
22
24
26
28
30
28
30
32
34
36
38
40
42
44

of the uniform distribution on page 557 with the pattern on page 561 of the parabolic to the 3rd power, we can see that the second sidelobe of the uniform distribution is located at the point of the first sidelobe of the parabolic. The alignment is better for n = 3 than for n = 2. The minimum sidelobes occur when the pedestal level is 27 dB below the peak of the distribution. A k space pattern of this distribution is plotted on page 564. From this plot we can see that the beamwidth with the pedestal is less than the case n = 3 plotted on page 561. The beamwidth factor with the pedestal is 1.493 compared to 1.651 without the pedestal. The first sidelobes of the uniform distribution are located within the main beam of the parabolic to the 3rd power distribution.

CIRCULAR TAYLOR DISTRIBUTION

Taylor has also published a technique for obtaining controlled sidelobes for a circularly symmetrical distribution on a circular aperture. The inner most zeros of the radiation pattern of a uniform circular aperture are replaced by new zeros which seeks to approximate the zeros of a Dolph-Tchebyscheff array and lower the sidelobes. The distribution in the aperture is expanded in terms of a series of Bessel functions.

The k space pattern of a circular aperture is given by the following integral.

$$f(k_\rho) = 2\pi \int_0^a E(\rho)\, J_0(k\rho \sin\theta)\, \rho \, d\rho$$

We will want to make a few substitutions so that the expression will match the literature.

$$u = 2a/\lambda \sin\theta \qquad p = \pi\rho / a \qquad g(p) = 2a^2 E(\rho)/\pi$$

With these substitutions the integral for the k space pattern of a circularly symmetric circular aperture becomes

$$f(\pi u) = \int_0^\pi p\, g(p)\, J_0(up)\, dp$$

The pattern of the uniform distribution (g = 1) is expressed in the new variables as

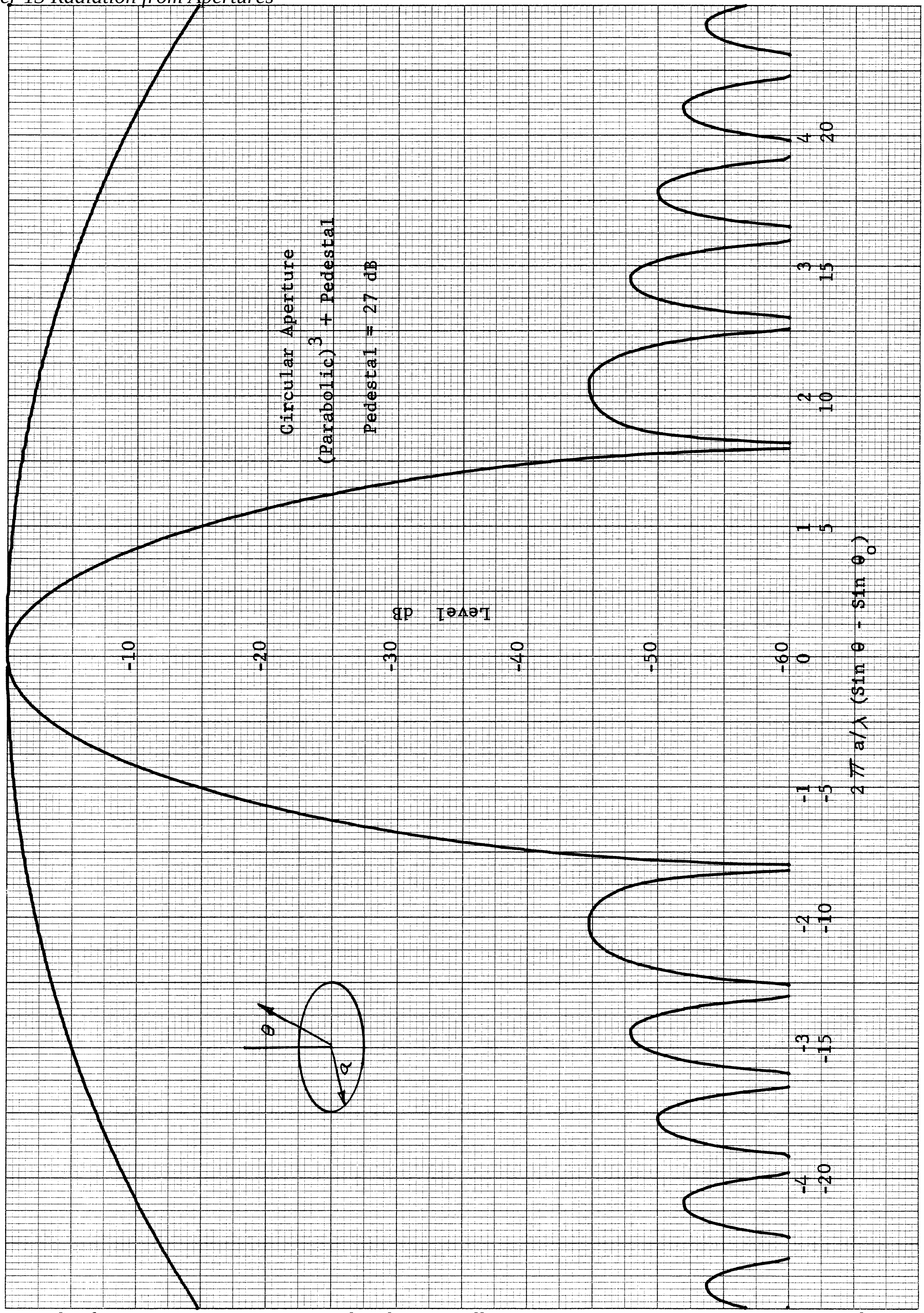
$$f(\pi u) = \frac{J_1(\pi u)}{\pi u}$$

This is similiar to the expression on page 543 in the discussion of the Taylor line source. We want to remove $\bar{n} - 1$ zeros of the radiation function.

$$J_1(X_{1n}) = 0 \qquad \text{Let } X_{1n} = \pi S_n \qquad n = 0, 1, 2, \ldots$$

X_{1n} are the zeros of $J_1(X)$. After we remove $\bar{n} - 1$ zeros of $J_1(x)$ and add back new zeros the radiation pattern in k space is given by the following expression.

$$\frac{J_1(\pi u)}{\pi u} \; \frac{\prod_{n=1}^{\bar{n}-1} (1 - u^2/u_n^2)}{\prod_{n=1}^{\bar{n}-1} (1 - u^2/S_n^2)}$$



46 1320

10 X 10 TO ½ INCH 7 X 10 INCHES
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The nulls at S_n were removed and new ones, U_n are added. Taylor gives the following expression for the new nulls.

$$U_N = S_{\bar{n}} \frac{(A^2 + (N - \frac{1}{2})^2)^{\frac{1}{2}}}{(A^2 + (\bar{n} - \frac{1}{2})^2)^{\frac{1}{2}}} \qquad N = 1, \ldots, \bar{n} - 1$$

This expression is the same as the one on page 547 except for a constant, $S_{\bar{n}}$. A is found from: $\cosh A = b$ with $20 \log_{10} b$ = Sidelobe level. Once we have the new nulls, we can find the radiation pattern.

The aperture distribution is expanded in a series of Bessel functions.

$$g(p) = \sum_{m=0}^{\infty} B_m J_0(S_m p)$$

Substituting the expression for g(p) in the radiation integral and reversing the order of the sum and the integral, we get

$$f(u) = \sum_{m=0}^{\infty} B_m \int_0^{\pi} p J_0(S_m p) J_0(up)\, dp$$

We will use point matching to find the coefficients, B_m. The point in the pattern at $U = S_k$ will only have a contribution from the k-th term in the sum.

$$f(S_k) = B_K \int_0^{\pi} p J_0^2(S_k p)\, dp$$

$$= B_K \left[\frac{p^2}{2} J_0^2(S_k p) + J_1^2(S_k p) \right|_0^{\pi}$$

The second term is zero at $p = 0$ and $p = \pi$ since $\pi S_k = X_{1k}$, a zero of $J_1(x)$. The first term is zero at $p = 0$.

$$f(S_k) = B_K \frac{\pi^2}{2} J_0^2(S_k \pi)$$

$$B_K = \frac{2 f(S_K)}{\pi^2 J_0^2(S_K \pi)}$$

The aperture distribution becomes

$$g(p) = \sum_{m=0}^{\bar{n}-1} B_m J_0(S_K p) \qquad p = \frac{\pi \rho}{a}$$

and the Fourier Bessel series only has $\bar{n}$ terms.

We are left with the problem of evaluating another indeterminate expression, $f(S_k)$, for the cofficients, B_k.

$$f(U) = d(u)/h(u) \qquad d(S_k) = h(S_k) = 0$$

h(u) is the same expression we had before on page 548.

$$h'(u) = \pi \prod_{N=1}^{\bar{n}-1} (1 - u^2/u_N^2) - 2u^2\pi \sum_{k=1}^{\bar{n}-1} \frac{1}{S_k^2} \prod_{\substack{N=1 \\ N\neq k}}^{\bar{n}-1} (1 - u^2/S_N^2)$$

$$h'(S_m) = -2\pi \prod_{\substack{N=1 \\ N\neq m}}^{\bar{n}-1} (1 - S_m^2/S_N^2)$$

The numerator, d(u), is only slightly different from the expression for the Taylor line source coefficients.

$$d'(u) = \pi J_1'(\pi u) \prod_{N=1}^{\bar{n}-1} (1 - u^2/u_N^2) - 2u J_1(\pi u) \sum_{k=1}^{\bar{n}-1} \frac{1}{S_k^2} \prod_{\substack{N=1 \\ N\neq k}}^{\bar{n}-1} (1 - u^2/u_N^2)$$

$$d'(S_m) = \pi J_1'(\pi S_k) \prod_{N=1}^{\bar{n}-1} (1 - S_m^2/u_N^2)$$

For m greater than zero we can use the following recurrence formula to find the derivative of the Bessel function.

$$J_1'(x) = J_0(x) - J_1(x)/x$$

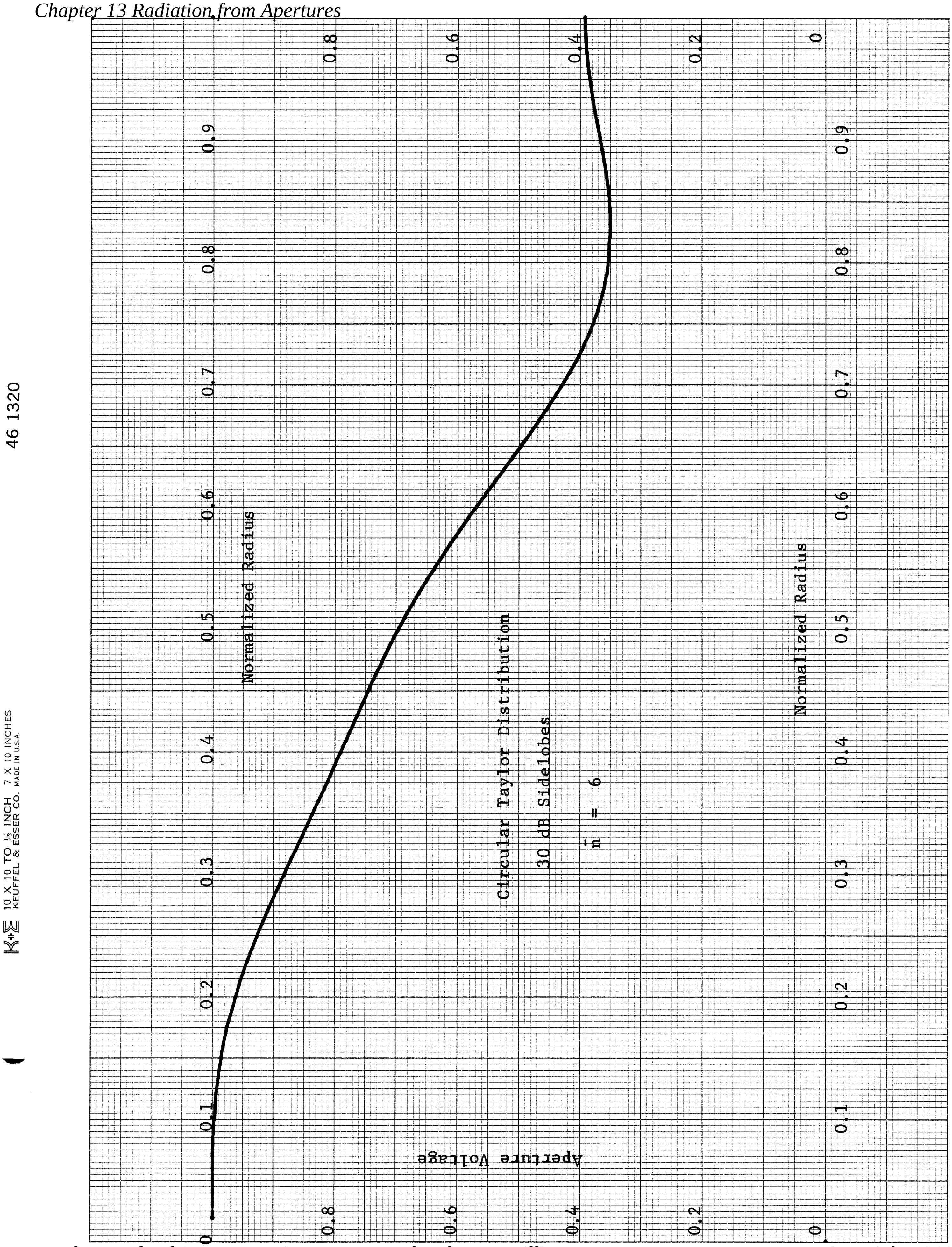
Since S_m is a zero of $J_1(x)$, the derivative of $J_1(x)$ evaluated at the zero of $J_1(x)$ is the Bessel function, $J_0(x)$. The expression for the coefficients becomes

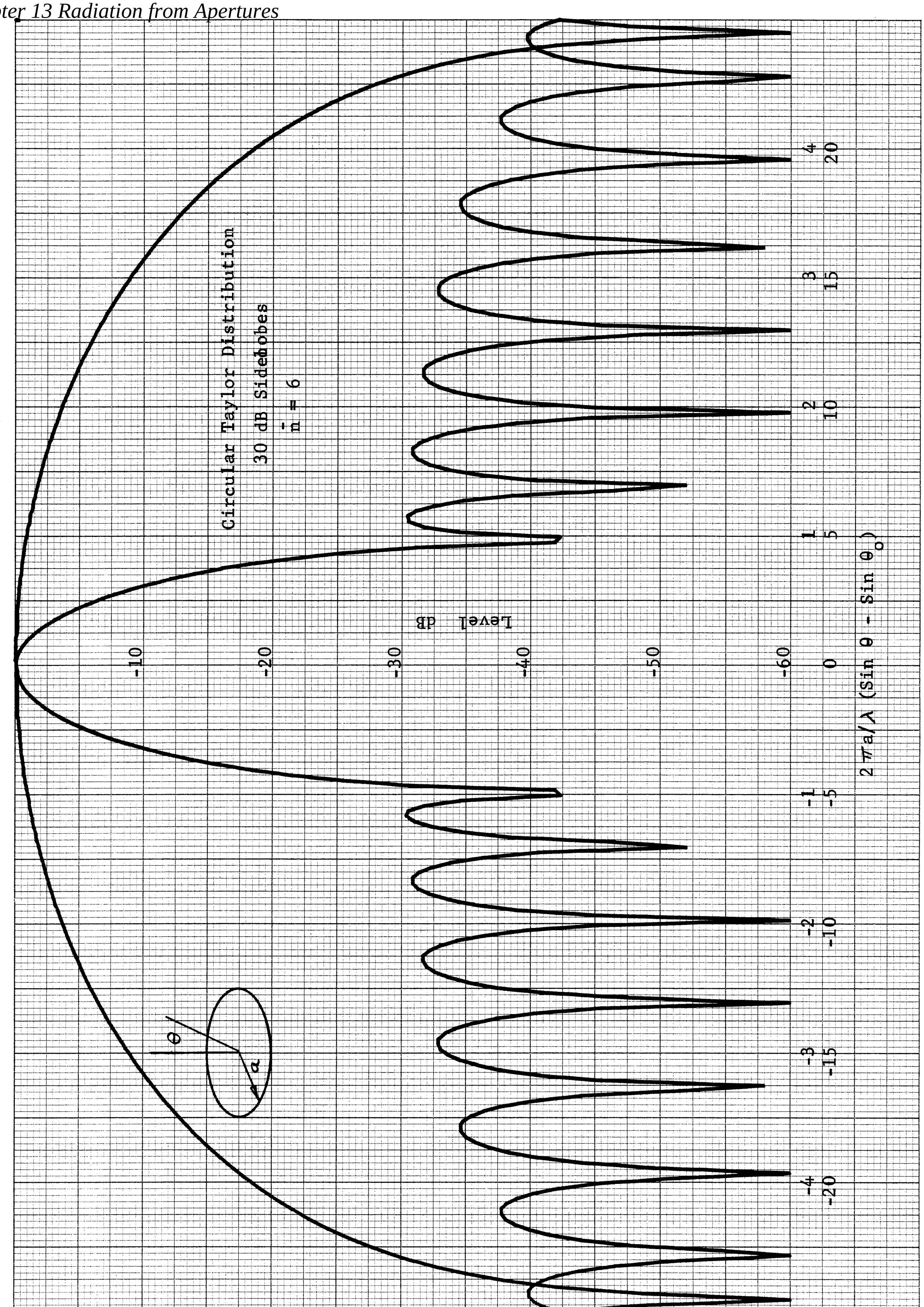
$$B_m = \frac{-\prod_{N=1}^{\bar{n}-1} (1 - S_m^2/u_N^2)}{\pi^2 J_0(\pi S_m) \prod_{\substack{N=1 \\ N\neq m}}^{\bar{n}-1} (1 - S_m^2/S_N^2)}$$

$$B_0 = 1./\pi^2$$

With these expressions the aperture distribution can be found for a given sidelobe level.

A Taylor distribution for a circular aperture was designed for a 30 dB maximum sidelobe level by modifying 5 zeros of the uniform circular aperture k space pattern or $\bar{n} = 6$. $\bar{n}$ is the first unmodified zero of the pattern. This is similiar to the example on page 548 of the Taylor line source. The result of the calculations is the aperture distribution given on page 567. If we compare this distribution to the Taylor line source distribution given on page 550, we see slight differences between the circular distribution and one-half of the linear distribution. The pattern for the distribution on page 567 is given on page 568. We see that the maximum sidelobe is 30 dB.


Circular Taylor Distribution
30 dB Sidelobes
$\bar{n} = 6$
Normalized Radius
Normalized Radius
Aperture Voltage
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


Circular Taylor Distribution
30 dB Sidelobes
$\bar{n} = 6$
Level dB
-10
-20
-30
-40
-50
-60
θ
a
$2\pi a/\lambda$ (Sin θ - Sin θ_o)
-4 -3 -2 -1 0 1 2 3 4
-20 -15 -10 -5 0 5 10 15 20

AMPLITUDE TAPER LOSS AND PHASE ERROR LOSS OF AN APERTURE

The directivity of a large aperture can be estimated from the beamwidths using the Kraus formula, page 35. The directivity can also be found from the electric field in the aperture when we assume the Huygens source approximation: the electric and magnetic fields are related by the intrinsic impedance of free space (see page 212). The directivity is found from the formula:

$$\text{Directivity} = \frac{\pi(1 + \cos\theta)^2}{\lambda^2} \frac{\left| \iint_S E(x,y)\, e^{j(k_x x + k_y y)}\, dx\, dy \right|^2_{max}}{\iint_S |E(x,y)|^2\, dx\, dy}$$

Suppose the beam maximum is at $\theta = 0$ (boresight), then the formula for directivity becomes

$$\text{Directivity} = \frac{4\pi}{\lambda^2} \frac{\left| \iint_S E(x,y)\, dx\, dy \right|^2}{\iint_S |E(x,y)|^2\, dx\, dy}$$

The maximum directivity occurs when the electric field has uniform amplitude and phase over the aperture. We would like to separate the loss in directivity due to amplitude and phase variations of the electric field in the aperture. Remember that the electric field, E(x,y) has both magnitude and phase (a complex number).

The directivity of the uniform aperture is given by

$$\text{Directivity} = 4\pi A/\lambda^2$$

where A is the area of the aperture. The directivity of the general aperture illumination can be expressed as

$$\text{Directivity} = (4\pi A/\lambda^2)(\text{ATL})(\text{PEL})$$

where ATL is the amplitude taper loss and PEL is the phase error loss.

Suppose the aperture distribution has uniform phase but an amplitude variation. The formula for directivity given above can be expanded in terms of the real and imaginary parts of the electric field phasor.

$$\left| \iint_S E_o\, dx\, dy \right|^2 = \left(\iint_S E_R\, dx\, dy \right)^2 + \left(\iint_S E_I\, dx\, dy \right)^2$$

E_R is the real part of E_o and E_I is the imaginary part. The integral in the denominator of the directivity formula can be separated also.

$$\iint_S |E_R|^2 + |E_I|^2\, dx\, dy$$

We have assumed that there are no phase variations in the aperture. Therefore, we can add an arbitrary phase to the electric field so that the field is only real with no effect on the directivity. The magnitude of the field is given by

$$E_o = (E_R^2 + E_I^2)^{\frac{1}{2}}$$

We can find the directivity for an aperture with uniform phase (PEL = 1) by the following formula.

$$\text{Directivity} = \frac{4\pi}{\lambda^2}\frac{\left|\iint_S |E_o|\,dx\,dy\right|^2}{\iint_S |E_o|^2 dx\,dy} = \left(\frac{4\pi A}{\lambda^2}\right)(ATL)$$

The phase has been changed until $E_I = 0$. We can solve for the ATL.

$$\text{Amplitude Taper Loss} = \frac{\left|\iint_S |E_o|\,dx\,dy\right|^2}{A\iint_S |E_o|^2 dx\,dy}$$

The formula is valid even when the phase error loss factor is not one, because by using only the magnitude of the electric field we have forced the phase response in the aperture to be uniform which is our assumption for the formula.

The phase error loss can be found from the following formula.

$$\text{PEL} = \frac{\text{Directivity}}{(\frac{4\pi}{\lambda^2}A)(\text{ATL})}$$

$$\text{Directivity} = \frac{4\pi}{\lambda^2}\frac{\left(\iint_S E_R\,dx\,dy\right)^2 + \left(\iint_S E_I\,dx\,dy\right)^2}{\iint_S |E_o|^2 dx\,dy}$$

We can identify the phase error loss by substituting the formula for directivity and ATL in the defining equation for PEL and cancelling terms.

$$\text{Phase Error Loss} = \frac{\left(\iint_S E_R\,dx\,dy\right)^2 + \left(\iint_S E_I\,dx\,dy\right)^2}{\left(\iint_S |E_o|\,dx\,dy\right)^2}$$

These formulas separate out the effects of amplitude variation losses and phase variations in the aperture on the directivity of the radiated pattern at boresight. If these losses are expressed in dB, then the directivity becomes:

$$\text{Directivity} = 10\,\text{Log}\,(4\pi A/\lambda^2) + (\text{ATL})_{\text{dB}} + (\text{PEL})_{\text{dB}}$$

The losses are relative to the uniform phase and amplitude aperture. It is important to notice that these are the losses at boresight. If there is a uniform phase taper across the aperture, then the beam will be squinted and the formula for phase error loss will predict the boresight loss. This could be quite large if boresight is in a null of the pattern. We can add an opposite linear phase taper across the aperture to bring the beam back to boresight. If we do this, then we are ignoring the beam scanning loss due to beam broadening (page 554). We can use this to separate out phase error loss due to quadratic or higher order phase error terms. These losses will become more important when we discuss parabolic reflector feeds.

Separable Aperture Distributions

We Spent a great deal of space discussing rectangular apertures whose distributions could be written as a product of two factors, each as a function of X or Y only. We need to develop formulas for the amplitude taper loss and phase error loss which can also be separated.

$$E_o(x,y) = E_1(x)\, E_2(y)$$

If we substitute this into the formula for amplitude taper loss, we get the following.

$$ATL = \frac{\left(\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} |E_1(x)E_2(y)|\, dx\, dy\right)^2}{ab\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} |E_1(x)\, E_2(y)|^2\, dx\, dy}$$

Where a b is the area of the rectangular aperture. We can separate these integrals.

$$ATL = \frac{\left(\int_{-a/2}^{a/2} |E_1(x)|\, dx\right)^2 \left(\int_{-b/2}^{b/2} |E_2(y)|\, dy\right)^2}{ab\int_{-a/2}^{a/2} |E_1(x)|^2 dx \int_{-b/2}^{b/2} |E_2(y)|^2 dy}$$

In this expression of integrals we can separate it into two parts in which each part only involves one coordinate.

$$ATL = (ATL)_x (ATL)_y$$

$$(ATL)_x = \frac{\left(\int_{-a/2}^{a/2} |E_1(x)|\, dx\right)^2}{a\int_{-a/2}^{a/2} |E_1(x)|^2 dx}$$

We get a similiar expression for $(ATL)_y$. We can use this to evaluate the amplitude taper loss of each of the linear separable aperture distributions considered before.

The uniform aperture distribution obviously has an amplitude taper loss of 1 (0 dB). The next distribution we considered was triangular. The aperture distribution is

$$E(x) = E_o(1 - |2x/a|) \qquad -a/2 \leq x \leq a/2$$

Note that this is an even function (symmetrical about the origin). We can evaluate each integral over half the interval and multiply it by two.

$$ATL_x = \frac{\left(2\int_0^{a/2}(1 - 2x/a)\,dx\right)^2}{2a\int_0^{a/2} 1 - 4x/a + 4x^2/a^2\,dx} = \frac{3}{4}$$

If we have an aperture with a triagnular distribution in both planes, then the amplitude taper loss will be $(3/4)^2$ = 2.5 dB.

We can list the amplitude taper loss for various linear distributions.

Distribution	Amplitude Taper Loss	
Uniform	1	0 dB
Triangular	3/4	1.25
Cosine	$8/\pi^2$	0.91
Cosine Squared	2/3	1.76
Cosine Squared on Pedestal $PD + (1 - PD)\cos^2(\pi x/a)$	$\frac{2(1+PD)^2}{3+5\,PD}$	
Pedestal		
10 dB	.756	1.21 dB
15	.713	1.47
20	.691	1.60

We can find the amplitude taper loss for various Taylor line sources by performing the indicated integrations on the aperture distributions.

Amplitude Taper Loss of Taylor Line Source

	Sidelobe Level						
$\bar{n}$	20	25	30	35	40	45	50
4	.17	.43	.69	.91			
6	.15	.39	.66	.92	1.15	1.35	
8	.16	.36	.63	.90	1.14	1.36	1.55
12	.24	.34	.59	.86	1.11	1.34	1.54
16	.35	.35	.57	.84	1.09	1.32	1.53
20	.46	.37	.56	.82	1.07	1.30	1.51

We can see that the amplitude taper loss of the Taylor line source is more a function of the sidelobe level than the number of terms in the Fourier Cosine series of the distribution ($\bar{n}$). Notice that there is an optimum number, $\bar{n}$, for each sidelobe level which can be seen in the cases for 20 and 25 dB max. sidelobe designs. In many cases we are interested in the minimum beamwidth. For those cases the maximum number $\bar{n}$ will give narrowest beamwidth.

The phase error loss can be separated into the product of factors for each axis if the aperture distribution function is also separable. The phase error loss (or efficiency) is given by the formula:

$$PEL = \frac{\left| \iint_S E \, dx\, dy \right|^2}{\left(\iint_S |E| \, dx\, dy \right)^2} \qquad E = E_1(x)\, E_2(y)$$

$$PEL = \frac{\left| \int_{-a/2}^{a/2} E_1(x)\, dx \int_{-b/2}^{b/2} E_2(y)\, dy \right|^2}{\left(\int_{-a/2}^{a/2} |E_1(x)|\, dx \int_{-b/2}^{b/2} |E_2(y)|\, dy \right)^2}$$

This can be separated into a factor for each axis.

$$PEL_x = \frac{\left| \int_{-a/2}^{a/2} E_1(x)\, dx \right|^2}{\left(\int_{-a/2}^{a/2} |E_1(x)|\, dx \right)^2}$$

The formula for $(PEL)_y$ is the same.

Quadratic Phase Error

Suppose we have separated aperture distribution factor which has a quadratic phase error and a uniform amplitude. We can define a term, S, which is the maximum phase error in wavelengths along the aperture. The aperture distribution is written below.

$$E(x) = e^{-j2\pi S(2x/a)^2}$$

The phase error is given by the formula for a single axis.

$$PEL_x = \frac{\left| 2 \int_0^{a/2} e^{-j2\pi S\left(\frac{2x}{a}\right)^2} dx \right|^2}{\left(2 \int_0^{a/2} \left| e^{-j2\pi S(2x/a)^2} \right| dx \right)^2}$$

The distribution is an even function so that we only need to take the interval over one half the aperture and multiply by 2.

Note that $|e^{-jx}| = 1$ which reduces the denominator to a^2. In the upper integral make the following substitution.

$$u = 4\sqrt{S}\,(x/a) \qquad du = \frac{4\sqrt{S}}{a}\,dx$$

$$(PEL)_x = \left| \frac{1}{2\sqrt{S}} \int_0^{2\sqrt{S}} e^{-j\frac{\pi u^2}{2}}\,du \right|^2$$

The integral must be divided into real and imaginary parts to find the magnitude.

$$(PEL)_x = \frac{1}{4S}\left[\left(\int_0^{2\sqrt{S}} \cos\left(\frac{\pi u^2}{2}\right) du \right)^2 + \left(\int_0^{2\sqrt{S}} \sin\left(\frac{\pi u^2}{2}\right) du \right)^2 \right]$$

$$= \frac{1}{4S}\left(C^2(2\sqrt{S}) + S^2(2\sqrt{S}) \right)$$

C(t) and S(t) are the Fresnel integrals. This function is plotted on page 575 along with other amplitude distributions. This graph shows that the more a distribution is tapered, the less susceptable it is to quadratic phase error loss. The edges of the pattern which have the most phase error are contributing the least to the pattern.

Example. On page 224 there are dimensions of a pyramidal rectangular horn which has quadratic phase error in both planes and the distribution is separable.

W = 11.38	(H plane width)	Frequency = 8 GHz
H = 8.38	(E plane height)	Wavelength = 1.475 in.
R_h = 20.06	(H plane slant length)	
R_e = 18.93	(E plane slant length)	

The maximum phase deviation in the aperture is found from the following formula.

$$S_e = \frac{H^2}{8\,R_e\lambda} \qquad S_h = \frac{W^2}{8\,R_h\lambda}$$

When we substitute the dimensions into these formula, we obtain: $S_e = 0.314$

$S_h = 0.547$

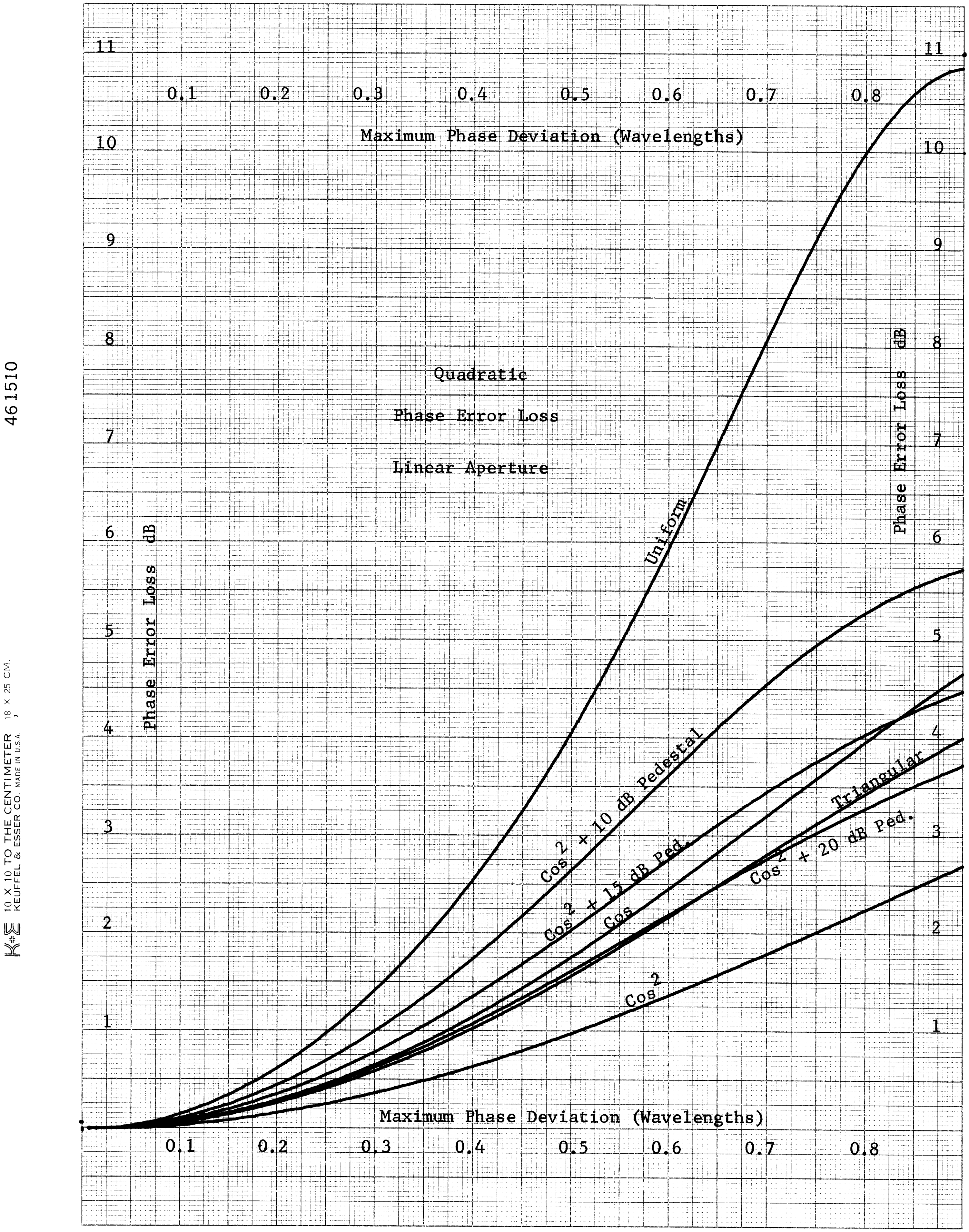
In the E plane the aperture distribution is uniform while the H plane has a Cosine distribution. We can read the following phase error losses off the curves on page 575.

$$(PEL)_e = 1.56 \text{ dB} \qquad (PEL)_h = 2.07$$

The amplitude taper loss of the uniform (E plane) distribution is 0 dB and the Cosine distribution (H plane) has a loss of 0.91 dB. The directivity of the horn is found from the expression:

$$\text{Directivity} = 10 \text{ Log } (4\pi A/\lambda^2) - (PEL)_e - (PEL)_h - (ATL)_h$$

The resulting directivity is 22.87 dB.


Maximum Phase Deviation (Wavelengths)
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
Quadratic
Phase Error Loss
Linear Aperture
Phase Error Loss dB
1 2 3 4 5 6 7 8 9 10 11
Uniform
Cos 2 + 10 dB Pedestal
Cos 2 + 15 dB Ped.
Cos
Triangular
Cos 2 + 20 dB Ped.
Cos 2
46 1510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.

Losses of Circularly Symmetrical Distributions

So many antennas are described by circularly symmetrical distributions that it will be worthwhile to derive formulas for them. The formulas that integrate over the surface of the aperture are still valid.

$$\text{Amplitude Taper Loss} = \frac{\left|\int_0^{2\pi}\int_0^a |E(r)|\, r\,dr\,d\phi'\right|^2}{\pi a^2 \int_0^{2\pi}\int_0^a |E(r)|^2 r\,dr\,d\phi'}$$

We can perform the ϕ' integrations simply since E(r) is not a function of ϕ'.

$$\text{Amplitude Taper Loss} = \frac{2\left|\int_0^a E(r) r\,dr\right|^2}{a^2 \int_0^a |E(r)|^2\, r\,dr}$$

Similiarly, we can find an expression for the phase error loss.

$$\text{Phase Error Loss} = \frac{\left(\int_0^a E_r(r)\, r\,dr\right)^2 + \left(\int_0^a E_I(r)\, r\,dr\right)^2}{\left(\int_0^a |E(r)|\, r\,dr\right)^2}$$

Let us find the amplitude taper loss of the parabolic distribution. We will normalize the radius of the aperture.

$$\text{ATL} = \frac{2\left|\int_0^1 (1 - r^2)^n\, r\,dr\right|^2}{\int_0^1 (1 - r^2)^{2n}\, r\,dr}$$

Make the substitution: $u = (1 - r^2)$; $du = -2\,r\,dr$

$$ATL = \frac{2\left|\frac{1}{2}\int_0^1 u^n\,du\right|^2}{\frac{1}{2}\int_0^1 u^{2n}\,du} = \frac{2n+1}{(n+1)^2}$$

Parabolic Distribution	Amplitude Taper Loss	
n	Ratio	DB
0	1	0
1	3/4	1.25
2	5/9	2.55
3	7/16	3.59

Parabolic on a Pedestal

$$\text{Amplitude Taper Loss} = \frac{(PD + (1 - PD)/(n + 1))^2}{PD^2 + \frac{2\ PD\ (1 - PD)}{n + 1} + \frac{(1 - PD)^2}{2\ n + 1}}$$

	Amplitude Taper Loss	
Pedestal	n = 2	n = 3
15 dB	1.12 dB	1.37 dB
20	1.61	2.09
25	1.97	2.64
27	2.08	2.82
30	2.21	3.02

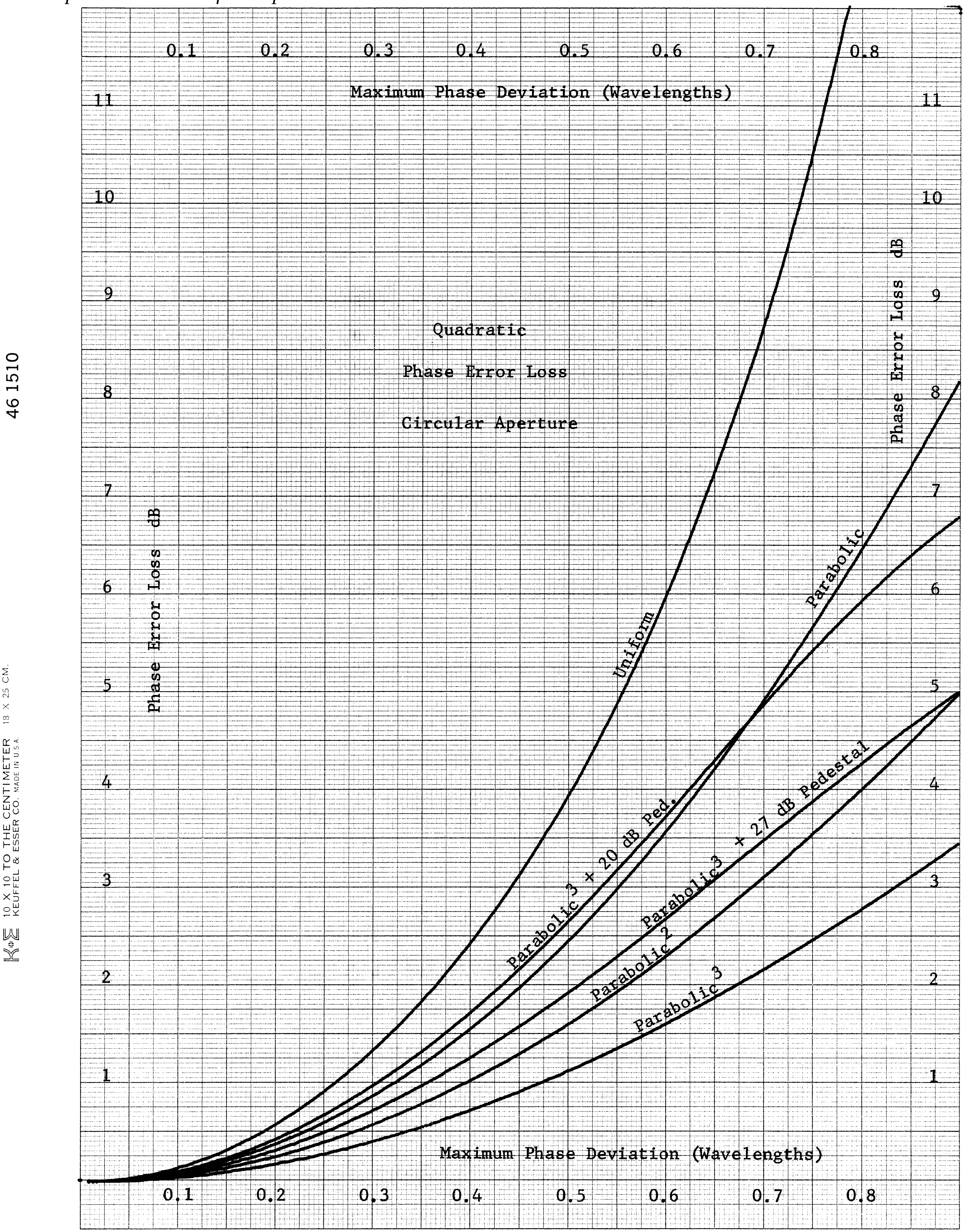
The table of amplitude taper losses for the parabolic^n on a pedestal shows that the loss is decreased by putting the distribution on a pedestal.

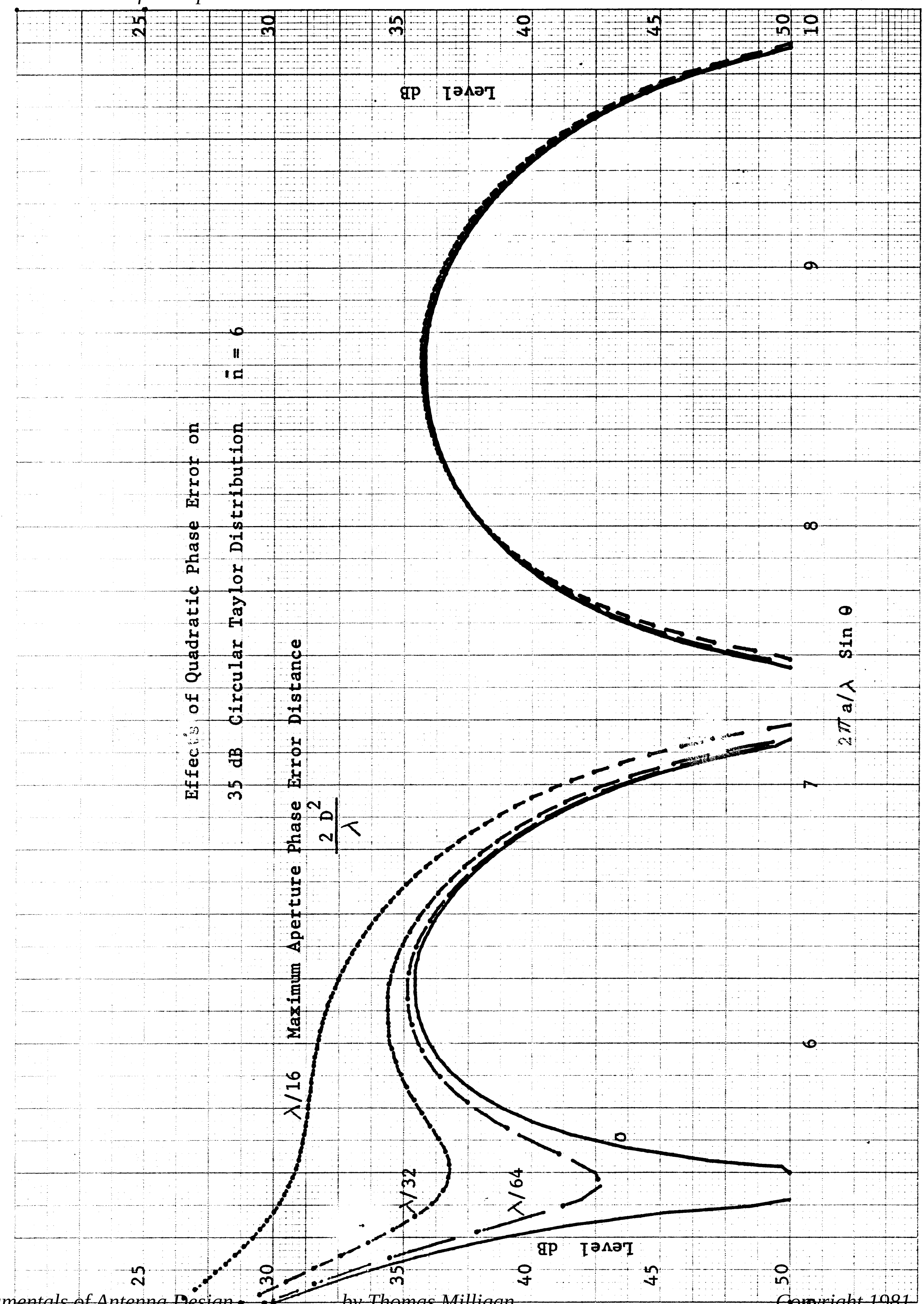
Quadratic phase errors will effect the circular aperture in a manner similiar to the rectangular aperture. A curve of the quadratic phase error loss is plotted on page 578 for various circularly symmetrical distributions. The uniform distribution is effected by phase errors more than the tapered distributions. Small quadratic phase errors will raise the sidelobes of low sidelobe antenna distributions. On page 579 is a plot of the first two sidelobes of a 35 dB circular Taylor distribution with small values of quadratic phase error. This has importance to antenna measurement. The source antenna would have to be spaced $8\ D^2/\lambda$ to measure the sidelobe level within 0.5 dB.

Amplitude Taper Loss of Circular Taylor Distribution

	Sidelobe Level dB					
$\bar{n}$	25	30	35	40	45	50
4	.30 dB	.71	1.14	1.51	1.84	
6	.28	.59	1.03	1.48	1.88	2.23
8	.43	.54	.94	1.40	1.82	2.21
12	1.03	.62	.86	1.28	1.71	2.12
16	1.85	.86	.87	1.22	1.64	2.05

This table is similiar to the one on page 572 for the linear Taylor distribution. The amplitude taper loss increases for decreased sidelobes. It also appears that there is an optimum $\bar{n}$ for each sidelobe level to minimize the amplitude taper loss, but it is a broad minimum.


0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Maximum Phase Deviation (Wavelengths)
Quadratic
Phase Error Loss
Circular Aperture
Phase Error Loss dB
Uniform
Parabolic
Parabolic 3 + 20 dB Ped.
Parabolic 3 + 27 dB Pedestal
Parabolic 2
Parabolic 3
1
2
3
4
5
6
7
8
9
10
11



46 1320

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