ARRAY SYNTHESIS

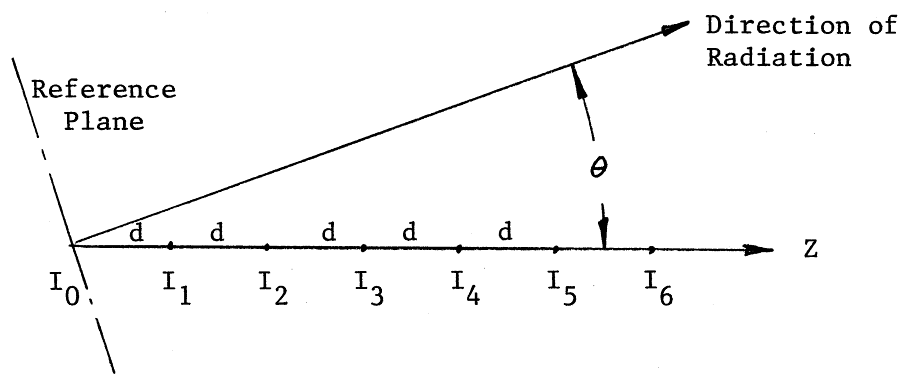
Chapter 5 is concerned with the analysis of arrays. In a sense synthesis can be performed by analyzing many different configurations until a mental pattern is established and synthesis becomes an efficient cut and try procedure. We must be able to accept this method. With computer codes available to analyze arrays easily, it is not reasonable to approach the problem in this way. If a mathematical description of the required array can be formulated, then optimization routines can design arrays or any other antenna by letting the computer search for a solution.

The first technique of synthesis given here is just a search technique which manipulates the zeros (or nulls) of the pattern. This is one step easier than adjusting the element phases and amplitudes. The other methods we will cover are more procedures moving from specifications to design.

SCHELKUNOFF'S UNIT CIRCLE METHOD

This method consists of manipulating the zeros of the array pattern to achieve a desired pattern. We can use the representation to describe any uniformly spaced array, although it is difficult to use it to design arrays with large numbers of elements. It is similiar to designing networks by manipulating the placement of poles and zeros in the complex plane. The array only has zeros to manipulate.

A linear uniformly spaced array was analyzed on page 105 with equal amplitudes. We can easily extend this to arrays with arbitrary feeding coefficients.



The array response will be symmetrical about the Z axis. If we define

$$\psi = k\, d \cos\theta + \delta$$

where δ is the fixed phase shift between elements, then the pattern of the array is given by

$$E = I_0 + I_1 e^{j\psi} + I_2 e^{j2\psi} + I_3 e^{j3\psi} + I_4 e^{j4\psi} + \ldots$$

I_i is the amplitude of each element in the array. It is a phasor quantity involving both amplitude and phase. We can further simplify the notation by defining:

$$W = e^{j\psi}$$

which reduces the array pattern to

$$E = I_0 + I_1W + I_2W^2 + I_3W^3 + I_4W^4 + \ldots$$

At this point the coefficients of the array feed are arbitrary. Remember that implicit in this discussion is that the patterns of all the elements are identical. In fact, we will only be concerned with isotropic radiators. The element pattern can be added later with pattern multiplication.

The array factor is given within an arbitrary constant by the following polynominal

$$f(W) = \sum_{n=0}^{N} a_n W^n$$

This polynominal has N roots which are the nulls of the pattern in W space. The coefficients, a_n, are complex which means that the zeros can be placed anywhere in the complex plane and not just in complex conjugate pairs as for real coefficients. The N roots are denoted: W_i. The array polynominal can be written as

$$f(W) = (W - W_1)(W - W_2)(W - W_3) \ldots (W - W_N)$$

We are not concerned with normalization. The amplitude of the array pattern is given by the magnitude of f(W).

$$|f(W)| = |W - W_1||W - W_2||W - W_3| \ldots |W - W_N|$$

$|W - W_i|$ is the distance from the root W_i to W in the complex plane. W has become the pattern variable. W is restricted to lying on the unit circle in the complex plane since

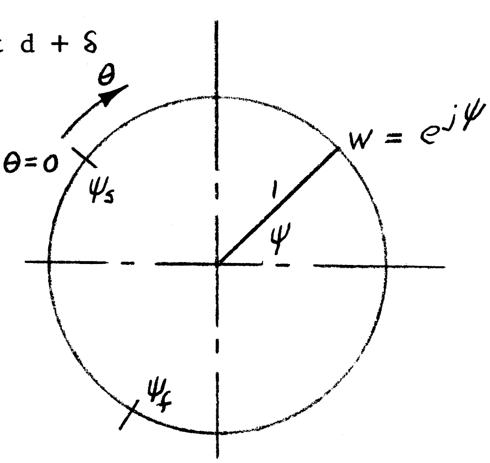
$$W = e^{j\psi}$$

which always has unit magnitude. The limits of W are determined by the spacing of the elements and the constant phase shift, δ, between elements.

$$\theta = 0 \qquad \psi_s = kd + \delta$$

$$\theta = 180° \qquad \psi_f = -kd + \delta$$

When θ increases, then ψ decreases.



There is no 2π limitation on either ψ_s or ψ_f. The number of times the unit circle is travelled is determined by the element spacing, 2 k d. ψ_s and ψ_f are the pattern limits in k space or the visible region. If $|\psi_s - \psi_f|$ exceeds 2π, then there will be a possibility of more than one main lobe (grating lobes). This is the same as the visible region on a polar diagram such as on page 107k where a grating lobe has appeared. From this we can see that it is only necessary that the visible region includes the beam maximum point twice for grating lobes.

The W space polynominal for a uniformly feed array can be obtained from page 105 by inspection

$$f(W) \doteq \frac{(1 - W^N)}{(1 - W)} \qquad \text{for N elements.}$$

The zeros of f(W) are the N zeros of $W^N = 1$ with the zero W = 1 removed.

$$W_i = e^{j2\pi i/N} \qquad i = 1, \dots , N - 1$$

These are uniformly spaced on the unit circle with one missing at $\psi = 0$.

Another special array is the binominal array discussed on page 113. The elements amplitudes are given by the binominal coefficients which means that the associated polynominal of the array is given by

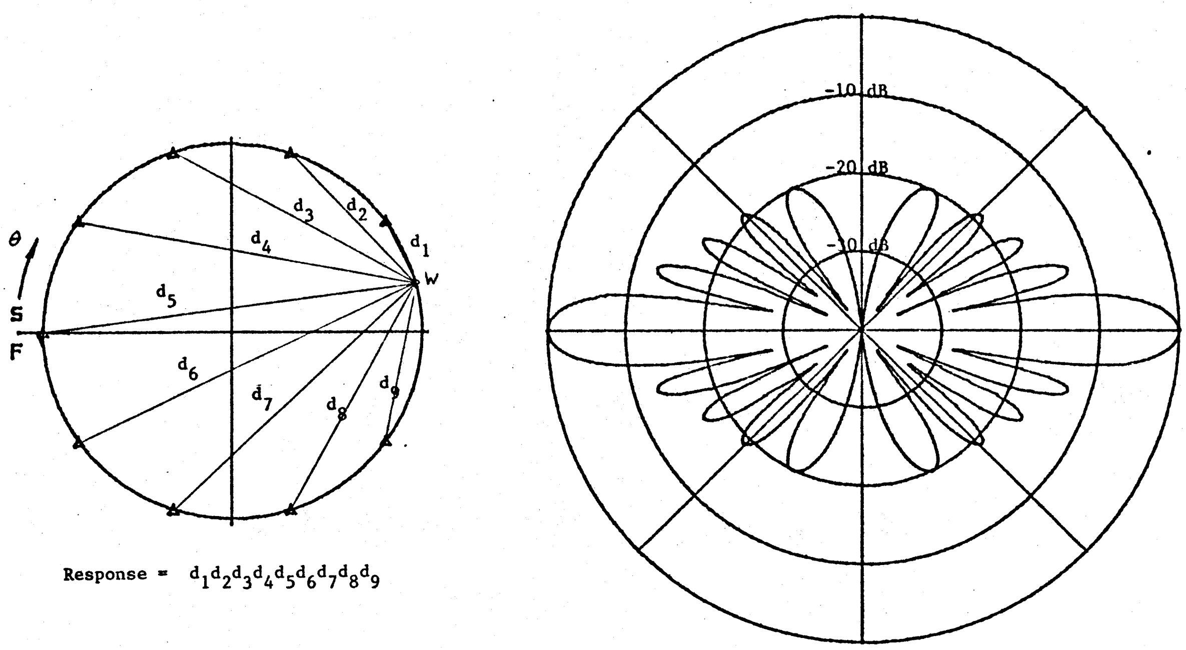
$$f(W) = (W + 1)^{N-1}$$

There are N-1 zeros at W = -1.

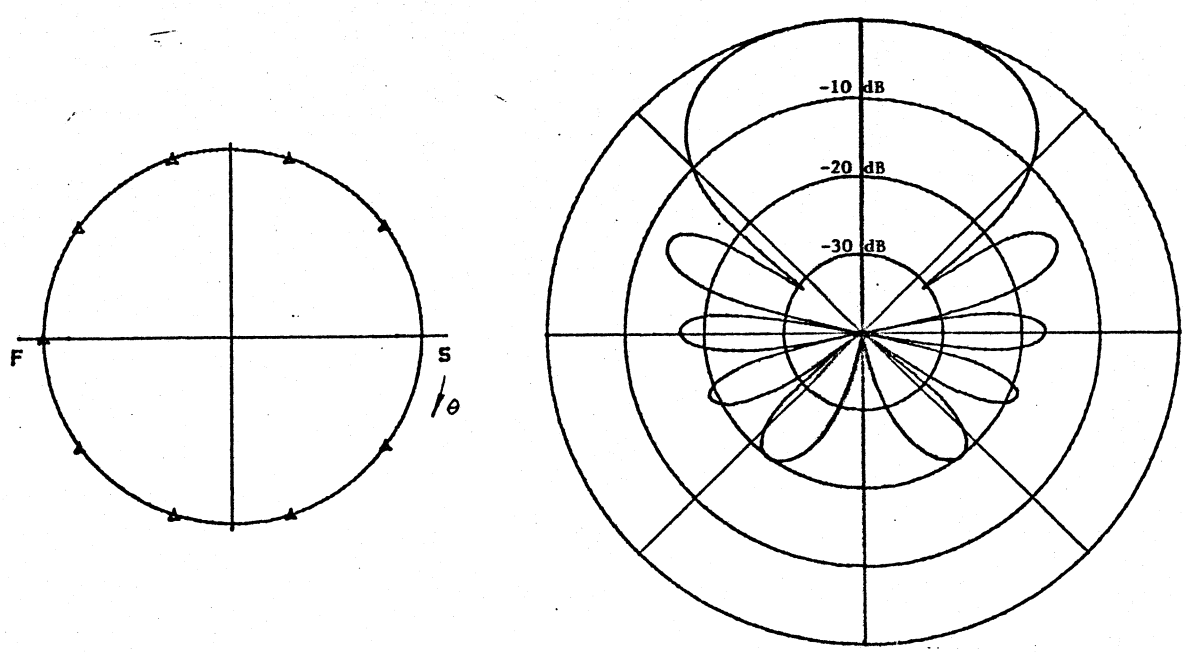
On the next few pages are examples of a few array patterns using zero diagrams. The pattern of a ten element array with spacings of 0.5 wavelengths and zero degrees phase between elements is drawn on the top of page 763. In the complex W plane, W starts at 180° and moves clockwise as θ increases. Moving from 180° to 0° on the unit circle we can see that 5 zeros are encountered which show as nulls in the pattern. Because the zeros are symmetrically placed on the unit circle, symmetrically placed nulls appear in the pattern from 90° to 180°. With half wavelength spacings the unit circle is completely traversed in the visible region.

The W plane diagram and pattern of an endfire array with quarterwave spacings is drawn on the bottom of page 763. The locations of the zeros have not changed from the diagram on the top of the page, but the start point and the distance traversed along the unit circle have changed. With quarterwave spacings between elements, the distance is only half-way around the unit circle. Without the -90° phase shift between elements the start point on the unit circle would be on the imaginary axis. Five zeros are encountered on the unit circle going from start to finish and appear in the pattern on the right. Of course, there is symmetry about the θ = 0 axis. The increased directivity endfire array (Hansen & Woodyard criterion) is obtained by changing the start and finish angles on the unit circle as shown on the top of page 764.

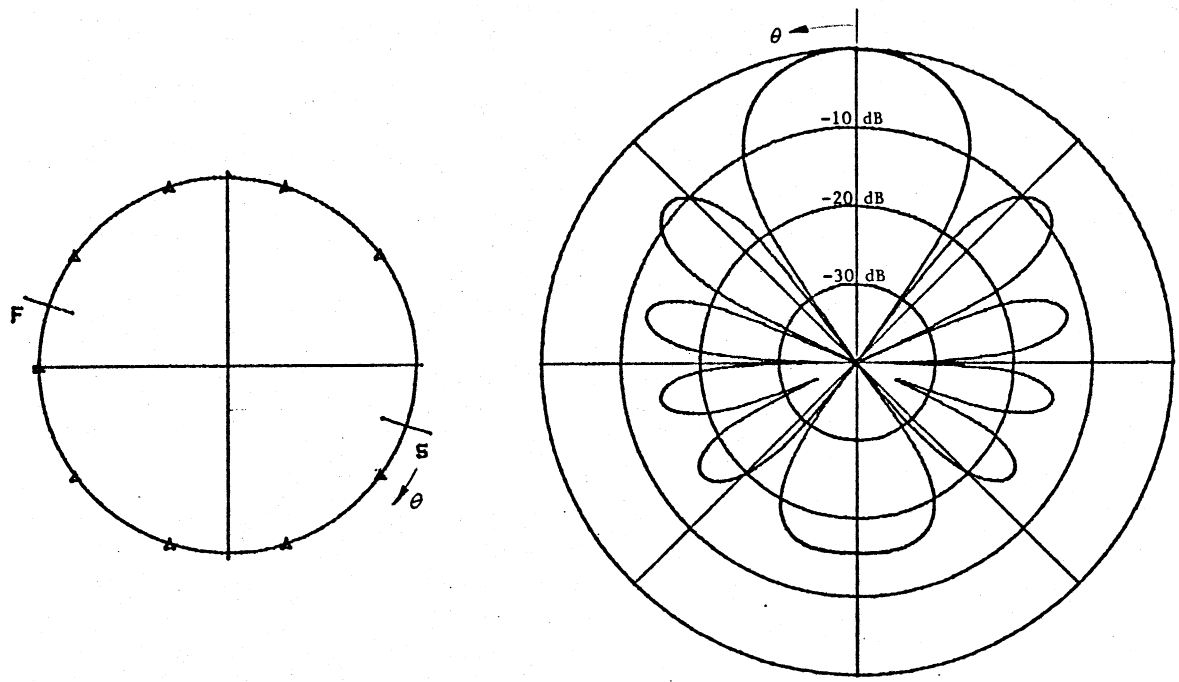
10 ELEMENT ARRAY 0.5 WAVELENGTH SPACINGS



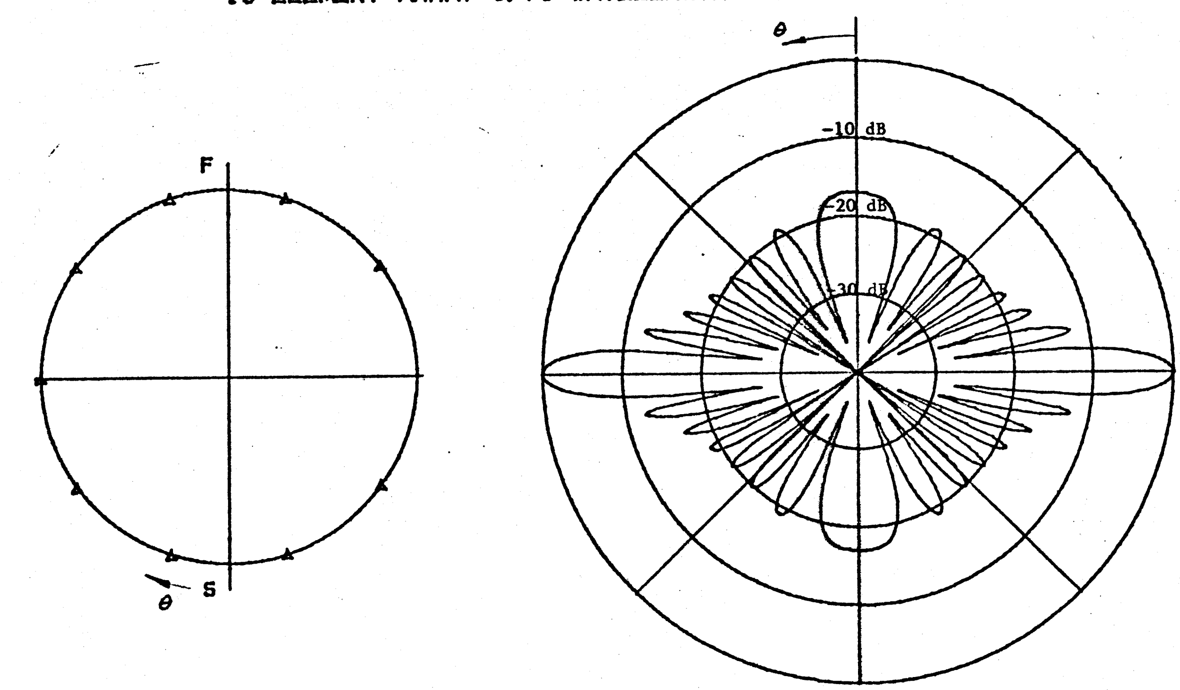
10 ELEMENT ARRAY 0.25 SPACINGS. -90 DEG. BETWEEN ELEMENTS



10 ELEMENT ARRAY 0.25 SPACING. −109 DEG BETWEEN ELEMENTS



10 ELEMENT ARRAY 0.75 WAVELENGTH SPACINGS

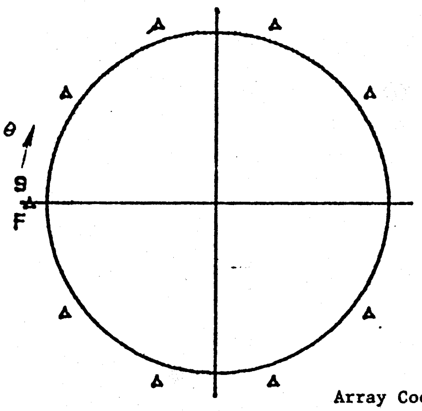


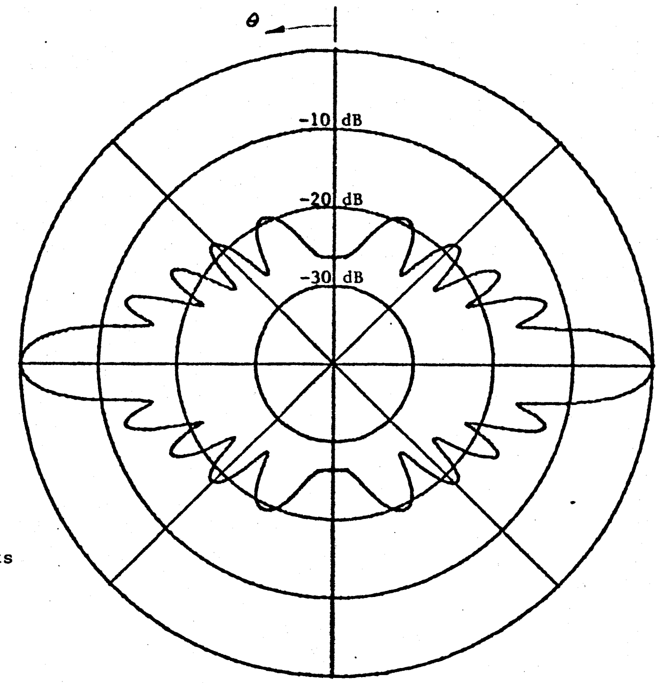
If the spacings between elements are greater than 0.5 wavelengths, then the unit circle is traversed further than once for $\theta = 0$ to $\theta = 180°$. The start point is on the negative imaginary axis for element spacings of 0.75 wavelengths and zero phase shift between elements. The pattern is generated as the point moves clockwise around the unit circle $1\frac{1}{2}$ times to the finish point. Because there is no phase shift between elements, S and F are symmetrically placed about the real axis. Referring to the diagram on the bottom of page 764, we can see that 14 zeros are encountered when moving from start to finish. These nulls appear in the pattern on the right. Although there are 14 nulls in the pattern, only 9 of them are independent. We can use these 9 zero locations, N - 1 for an N element array as variables to shape the pattern. It is not necessary to limit the zero locations to the unit circle. On the top of page 766 is a pattern whose zeros have the same angles as the uniform array but have a magnitude of 1.1. We can expect that there would not be any nulls in the pattern which is seen in the pattern on the right. When we multiply out the factors of the polynominal, we obtain the coefficients of the polynominal which are the feeding coefficients of the array elements. The magnitude of the coefficients are no longer symmetrical about the center of the array which is seen in the listing of the coefficients.

We can change the sidelobe levels or remove lobes in particular directions by manipulating the locations of the zeros in the W plane. The sidelobes in a region of space can be reduced by moving the zeros closer together. Any time we do this, we can expect the main beam to widen or some other sidelobes to become greater as the zeros are removed from these areas. On the bottom of page 766 the zeros of a 10 element array have been moved toward the negative real axis relative to uniform feed locations. This reduces the sidelobe levels as the zeros move toward the binominal array (no sidelobes). The peak sidelobe is down to 30 dB but the main beamwidth has increased from the pattern on the bottom of page 764. We could decrease the main beamwidth by moving the zeros until the sidelobes near $\theta = 0°$ and $\theta = 180°$ were also 30 dB. Because the zeros are placed symmetrically about the real axis, the coefficients of the array are only real. The feed coefficients of the array are tapered toward the ends. Since we can think of the array as a sampled aperture, the results of aperture will hold in part here. On page 530 we discussed the effects of high harmonic terms in the Fourier transforms which show as sidelobes. The greater the taper and the lack of discontinuities result in low sidelobes. An array is analyzed by Fourier series and the results from Fourier transforms carry over. Hence the taper toward the ends of the array results in low sidelobes.

We can shift zeros to place nulls in the pattern at selected points. Consider the pattern of the endfire array on the bottom of page 763. Suppose we want to eliminate the lobe at $\theta = 90°$. There are four zeros in invisible space. We can move one of these zeros to the negative imaginary axis which corresponds to $\theta = 90°$. The three remaining zeros in invisible space are moved until they are symmetrical. The resulting zero diagram and pattern are given on the top of page 767. The lobe at $\theta = 90°$ has been removed and the nearby sidelobes have been reduced. Even the beamwidth has been decreased. We can get further reductions in beamwidth by moving more of the zeros into visible space. We can obtain the pattern given on the bottom of page 767

10 ELEMENT ARRAY 0.5 WAVELENGTH SPACINGS

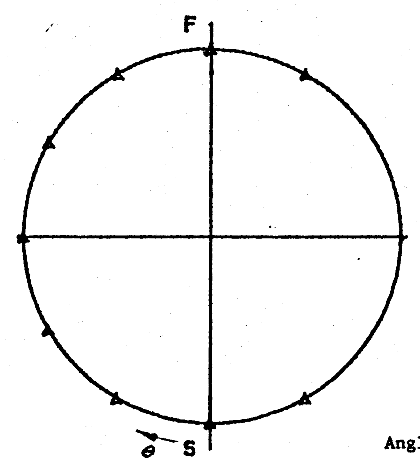


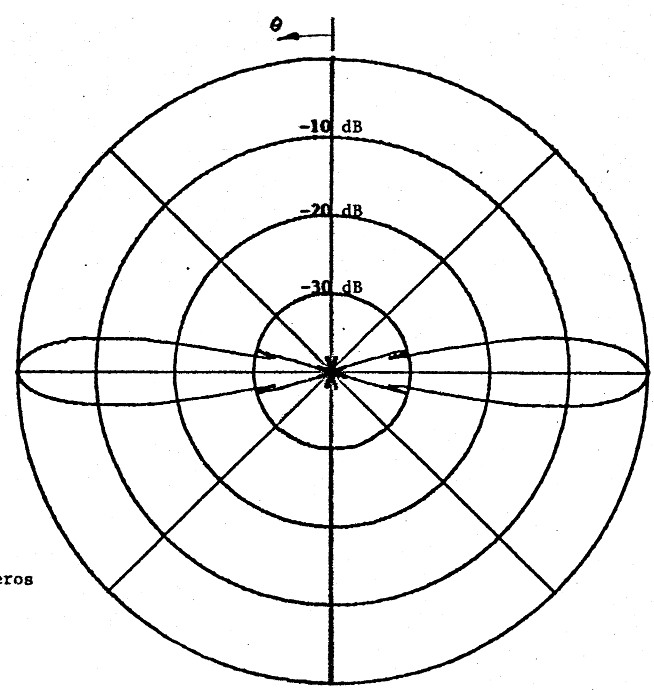


Array Coefficients

1	-6.91 dB
2	-7.73
3	-8.56
4	-9.39
5	-10.22
6	-11.05
7	-11.87
8	-12.70
9	-13.53
10	-14.36

10 ELEMENT ARRAY 0.75 WAVELENGTH SPACINGS





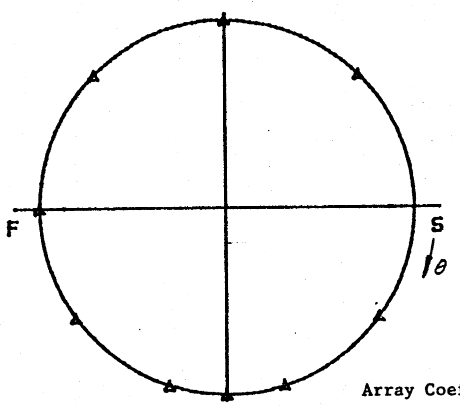
Array Coefficients

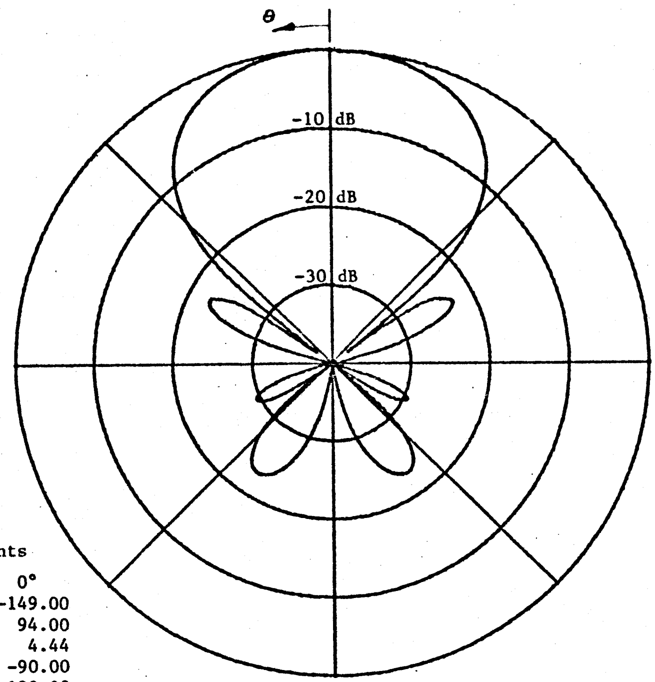
1	-24.09 dB
2	-15.36
3	-10.59
4	-7.88
5	-6.63
6	-6.63
7	-7.88
8	-10.59
9	-15.36
10	-24.09

Angles of Zeros

1	60
2	90
3	120
4	150
5	180
6	-150
7	-120
8	-90
9	-60

10 ELEMENT ARRAY 0.25 SPACINGS. −90 DEG. BETWEEN ELEMENTS





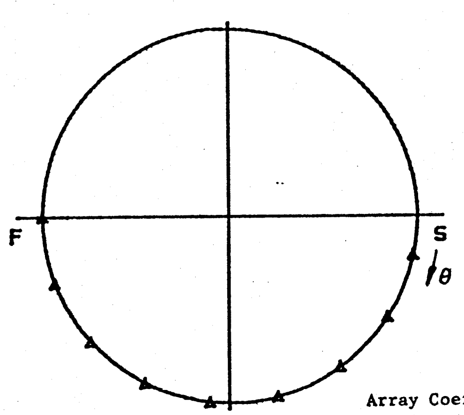
Angles of Zeros

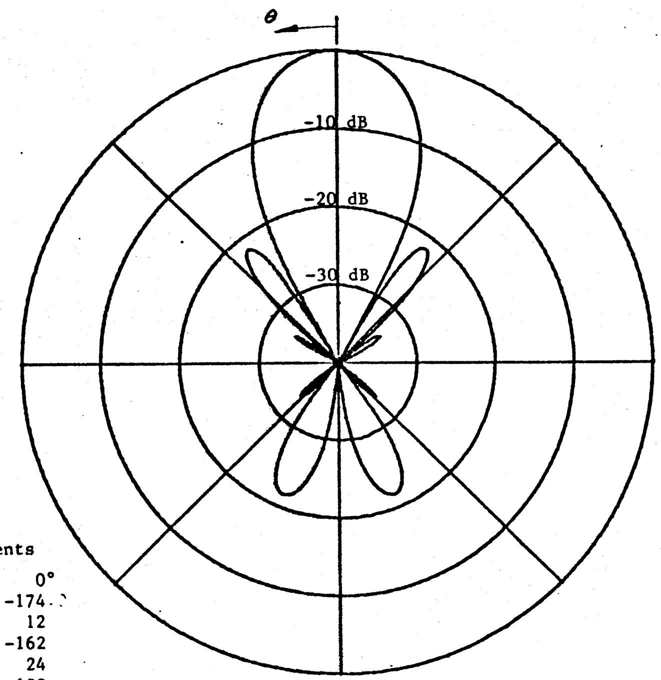
1	45°
2	90
3	135
4	180
5	-144
6	-108
7	-90
8	-72
9	-36

Array Coefficients

1	-13.82 dB	0°
2	-8.06	-149.00
3	-9.38	94.00
4	-10.28	4.44
5	-10.31	-90.00
6	-10.31	180.00
7	-10.28	85.56
8	-9.38	-4.00
9	-8.06	-121.00
10	-13.82	90.00

10 ELEMENT ARRAY 0.25 SPACINGS. −90 DEG. BETWEEN ELEMENTS





Angles of Zeros

1	-12
2	-33
3	-54
4	-75
5	-96
6	-117
7	-138
8	-159
9	180

Array Coefficients

1	-36.84 dB	0°
2	-22.08	-174.[illegible]
3	-13.21	12
4	-7.94	-162
5	-5.45	24
6	-5.45	-150
7	-7.94	36
8	-13.21	-138
9	-22.08	48
10	-36.84	-126

when we move all the zeros into visible space. If we look at the angles between elements, we see that they are near 180°. This means that the mutual coupling between elements is quite high. The input impedance of each antenna will be quite low and the stored energy will be high. There is a large beam in invisible space because there are no zeros. This large beam is stored energy and will greatly decrease the efficiency. On page 93 we discussed a two element version of an odd mode array which also shows high directivity but low efficiency.

This technique of moving zeros in the W plane is only suitable for arrays with only a few elements. It can be used for any size array except that it becomes unwieldly. One of the best uses is to find the feeding coefficients for broadcast towers to give a desired pattern. It is necessary to put nulls in these patterns to prevent interference between stations using the same band. Kraus gives an example of one of these requirements: a pattern must have uniform coverage between ±45° about north ($\theta = 0$) and nulls due east ($\theta = 270°$) and SW ($\theta = 135°$). Kraus solved this problem by using pattern multiplication, but we will use the Schelkunoff unit circle method. We will place the axis of the array along the N-S axis which will give us a symmetrical pattern about this axis. Since there are only two required nulls, we can meet this with 3 antennas. We will set $\delta = -90°$ to get an endfire antenna array for elements spaced a quarterwave apart. The actual phases between the elements will be determined from the zeros as well. Suppose a null in the pattern is required at θ_n. The zero in the W plane required to give this is given by

$$W_n = e^{j(2\pi/\lambda\ S \cos\theta_n + \delta)}$$

We need a null at 90°: $W_1 = e^{j\delta} = -j$ since $\delta = -90°$

The other null gives us the angle of the second zero.

$$(360°)\ \tfrac{1}{4} \cos(135°) - 90° = -153.64°$$

Both zeros are on the unit circle. We can find the array feeding coefficients by expanding the zero representation to the polynominal representation.

$$(W - e^{-j\,90°})(W - e^{-j\,153.64°})$$

$$(W + j)(W + 0.896 + j\,0.444)$$

$$W^2 + (0.896 + j\,1.444)\,W + (-0.444 + j\,0.896)$$

$$W^2 + 1.6994\,e^{j\,53.18°} + e^{j\,116.36°}$$

We will normalized to the phase of the first element (constant term of the polynominal).

Kraus, J. Antennas, pp. 69, McGraw Hill, New York, 1950.

$$W^2 e^{-j\,116.36°} + 1.6944\,W\,e^{-j\,58.18°} + 1$$

At this point the polynominal respresents an unscanned array ($\delta = 0$). We must add the - 90° phase shift between elements.

$$W^2 e^{-j\,296.36°} + 1.6944\,W\,e^{-j\,148.18°} + 1$$

The phase shift (-90°) is added to the second element and twice the phase shift is added to the third. The coefficients are the voltage feeding weights of the array. If we normalize the input power to one and renormalize the phase to the center element to demonstrate the symmetry, then we get the coefficients given on the top of page 770 where the W plane diagram and pattern are drawn.

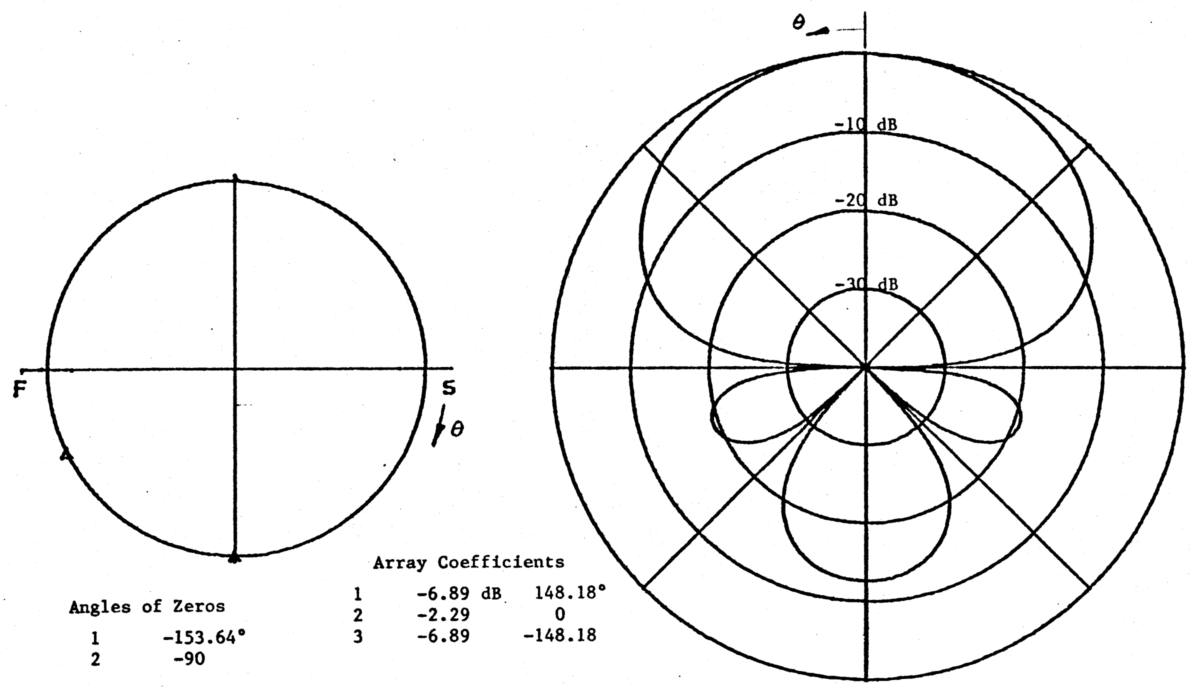
We can improve the pattern by adding an extra element. We will place an additional null in the pattern at $\theta = 180°$. The first two zeros in the W plane are in the same places as the 3 element array. To get a null in the pattern at $\theta = 180°$, we place a zero on the negative real axis. The W space diagram and pattern of this array are drawn on the bottom of page 770. There is however no improvement in the flatness of the response between ± 45°. On page 771 are two diagrams with the elements spaced further apart than $\lambda/4$. The distance around the unit circle from start to finish increases as the spacings increase. In order for the start to be on the real axis, the phase angles between the elements must be increased: - 108° for 0.3 wavelength spacings and - 144° for 0.4 wavelength spacings. It appears that a spacing half-way between the two patterns given on page 771 would give the flattest response. Both the dB and voltage patterns are drawn on page 772 and show almost equal ripple response over ± 45°.

DOLPH TCHEBYSCHEFF LINEAR ARRAY

Using the Schelkunoff unit circle method we can notice a number of things. Only zeros on the unit circle contribute to nulls in the pattern which means we get the best results by keeping them there. It appears that no matter where the zero are placed on the unit circle, the magnitude of the array coefficients are symmetrical about the center line of the array. From algebra we know that in order to have all elements of a broadside array in phase, the zeros of the polynominal must occur in complex conjugate pairs. As we move the zeros toward the negative real axis, the sidelobes are reduced. Finually we have a binominal array which has all zeros at -1 and no sidelobes. When we move the zeros toward the negative real axis, the main beam broadens. The minimum beamwidth occurs when all the sidelobes are the same level. To design an array, we could move the zeros around until all the sidelobes are the same. For a small array we could get a solution after just a few tries, but as the number of elements increased, we would take more and more time to reach a solution. We need a generalized method of finding the zero positions to give uniform sidelobes for a given sidelobe level and number of elements.

Dolph realized that the Tchebyscheff polynominals possess the property that they have equal ripples between -1 and 1 and then rise rapidly outside this region. Like most special functions, these are solutions to a differential

3 ELEMENT ARRAY 0.25 SPACINGS. -90 DEG BETWEEN ELEMENTS



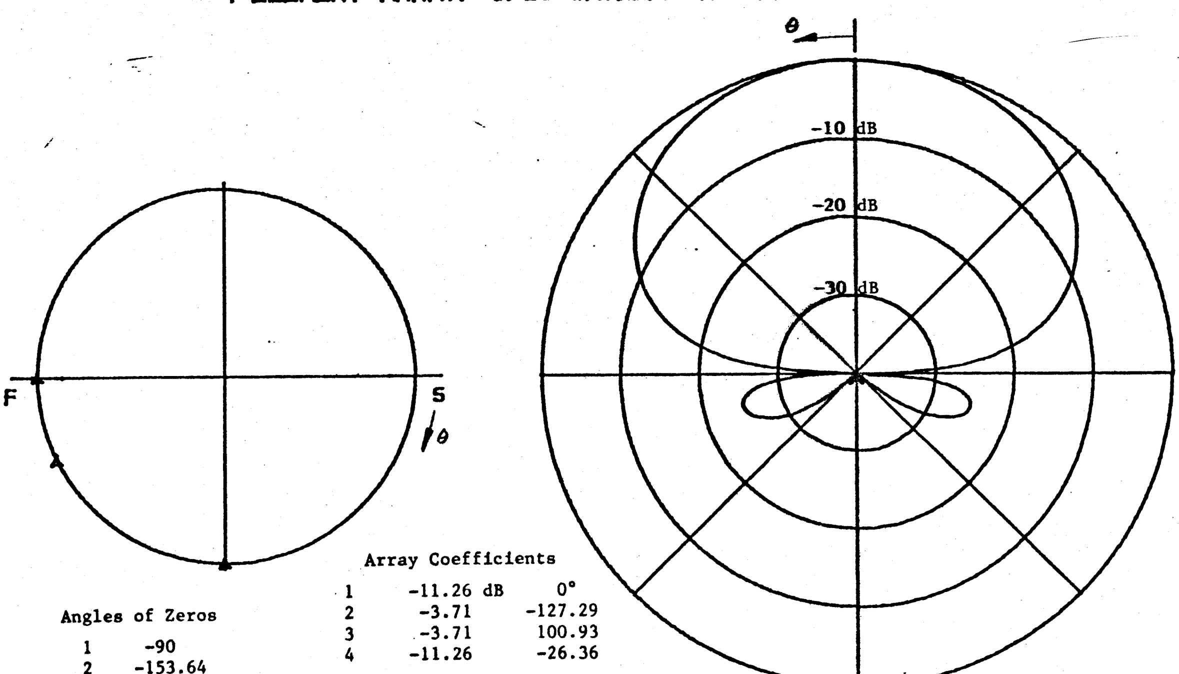
Array Coefficients

1	-6.89 dB	148.18°
2	-2.29	0
3	-6.89	-148.18

Angles of Zeros

1	-153.64°
2	-90

4 ELEMENT ARRAY. 0.25 WAVELENGTH SPACINGS



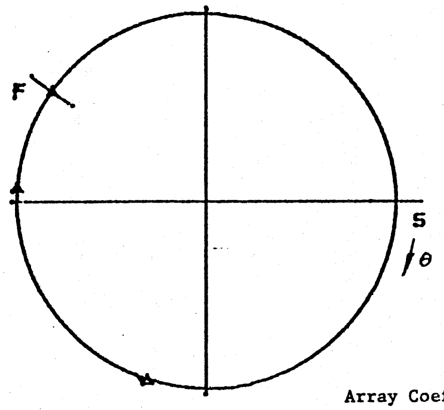
Array Coefficients

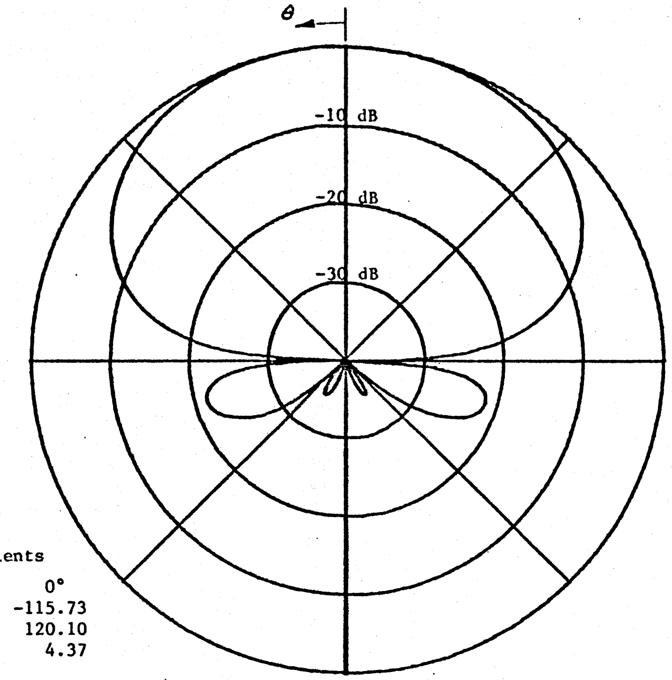
1	-11.26 dB	0°
2	-3.71	-127.29
3	-3.71	100.93
4	-11.26	-26.36

Angles of Zeros

1	-90
2	-153.64
3	180

4 ELEMENT ARRAY, 0.3 WAVELENGTH SPACINGS





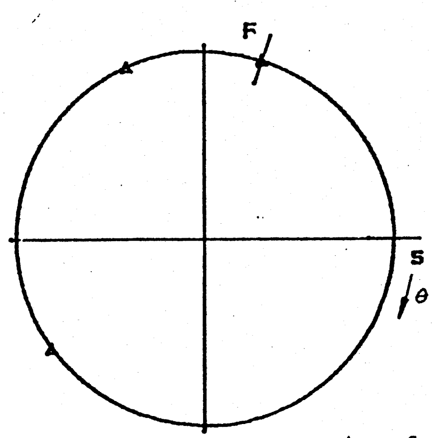
Angles of Zeros

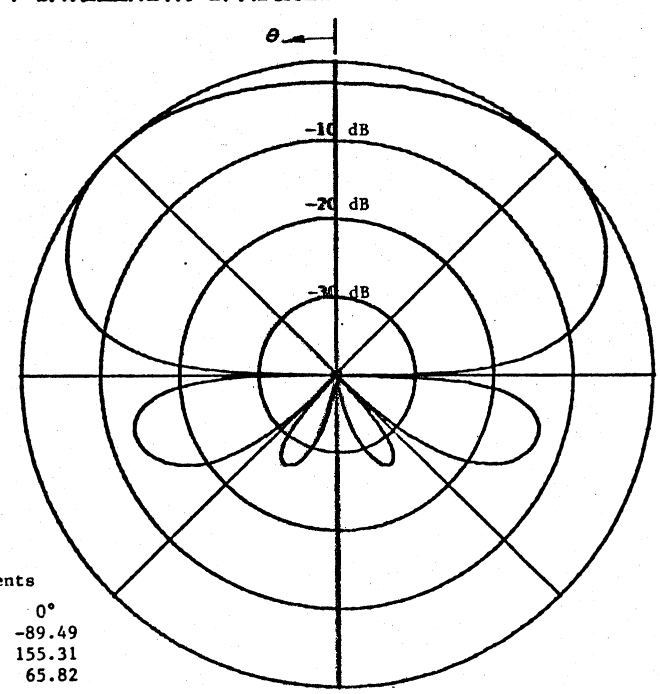
1	-108
2	175.63
3	144

Array Coefficients

1	-10.46 dB	0°
2	-3.87	-115.73
3	-3.87	120.10
4	-10.46	4.37

4 ELEMENT ARRAY, 0.4 WAVELENGTH SPACINGS





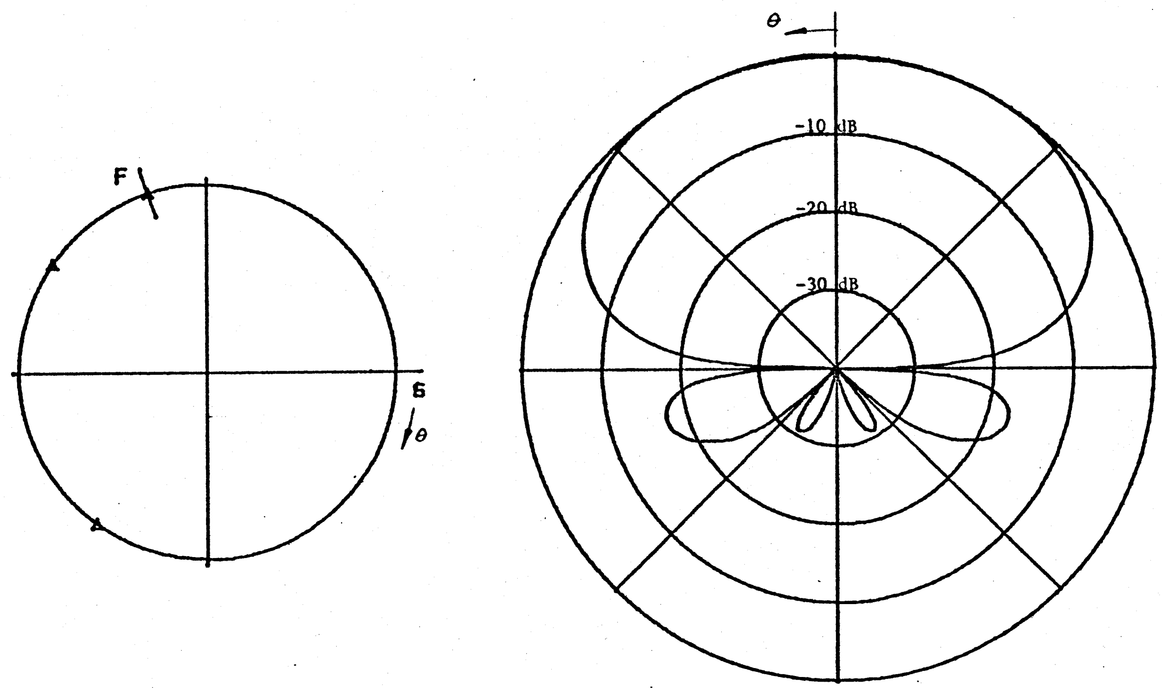
Angles of Zeros

1	-144°
2	114.18
3	72.0

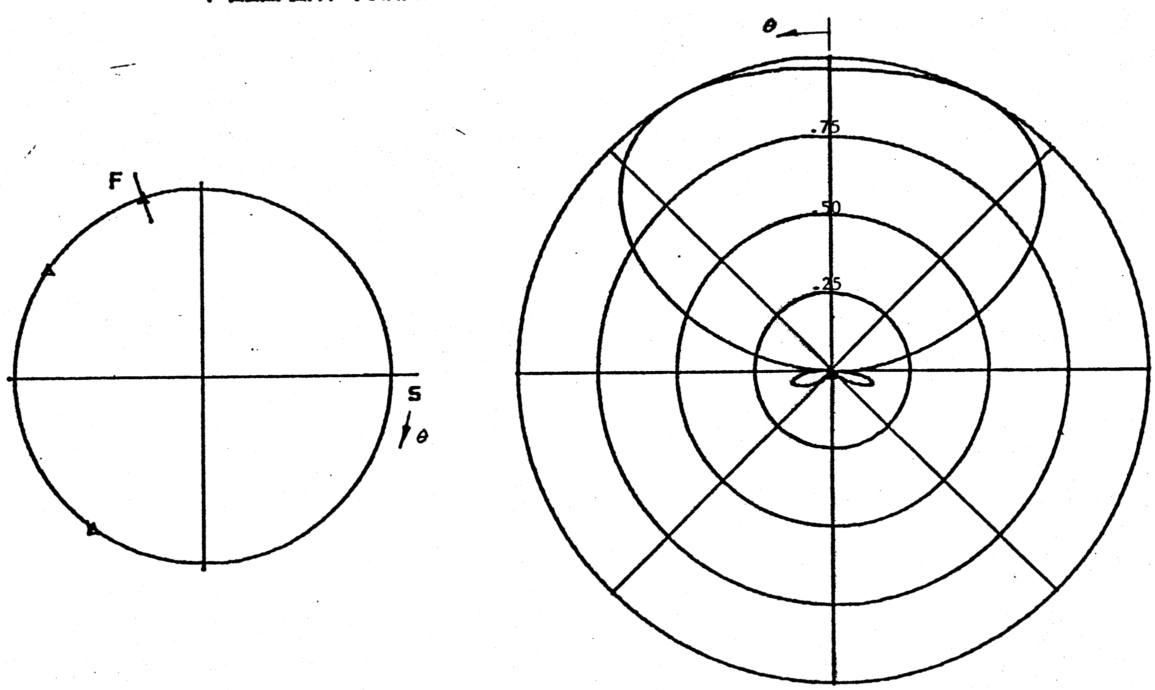
Array Coefficients

1	-8.39 dB	0°
2	-4.49	-89.49
3	-4.49	155.31
4	-8.39	65.82

4 ELEMENT ARRAY. 0.35 WAVELENGTH SPACINGS



4 ELEMENT ARRAY. 0.35 SPACINGS. VOLTAGE PATTERN



equation.

$$(1 - x^2)\frac{d^2T_m(x)}{dx^2} - x\frac{dT_m(x)}{dx} + m^2 T_m(x) = 0$$

There are different solutions for each integer m and consist of polynominals

$$T_0(x) = 1 \qquad T_1(x) = x \qquad T_2(x) = 2x^2 - 1 \qquad T_3(x) = 4x^3 - 3x$$

These polynominals are related by a recursion formula:

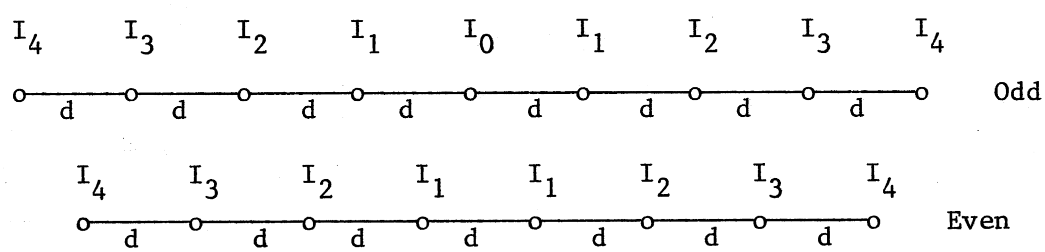
$$T_{m+1}(x) = 2\,x\,T_m(x) - T_{m-1}(x)$$

For our purposes it is more useful to use the form

$$T_m(x) = \begin{cases} (-1)^m \operatorname{Cosh}(m \operatorname{Cosh}^{-1}|x|) & x < -1 \\ \operatorname{Cos}(m \operatorname{Cos}^{-1} x) & -1 \le x \le 1 \\ \operatorname{Cosh}(m \operatorname{Cosh}^{-1} x) & x > 1 \end{cases}$$

The order of the Tchebyscheff polynominal is also the number of roots. Some of the lower ordered polynominals are plotted on page 774. Outside of the region $-1 \le x \le 1$, the polynominal value rises rapidly. Dolph devised a method of relating the Tchebyscheff polynominals to the array factor polynominal for a broadside array. The equal ripple portion is related to the sidelobes while the exponential increase beyond x = 1 is related to the main beam.

Consider a broadside array whose magnitudes are symmetrical about the center axis.



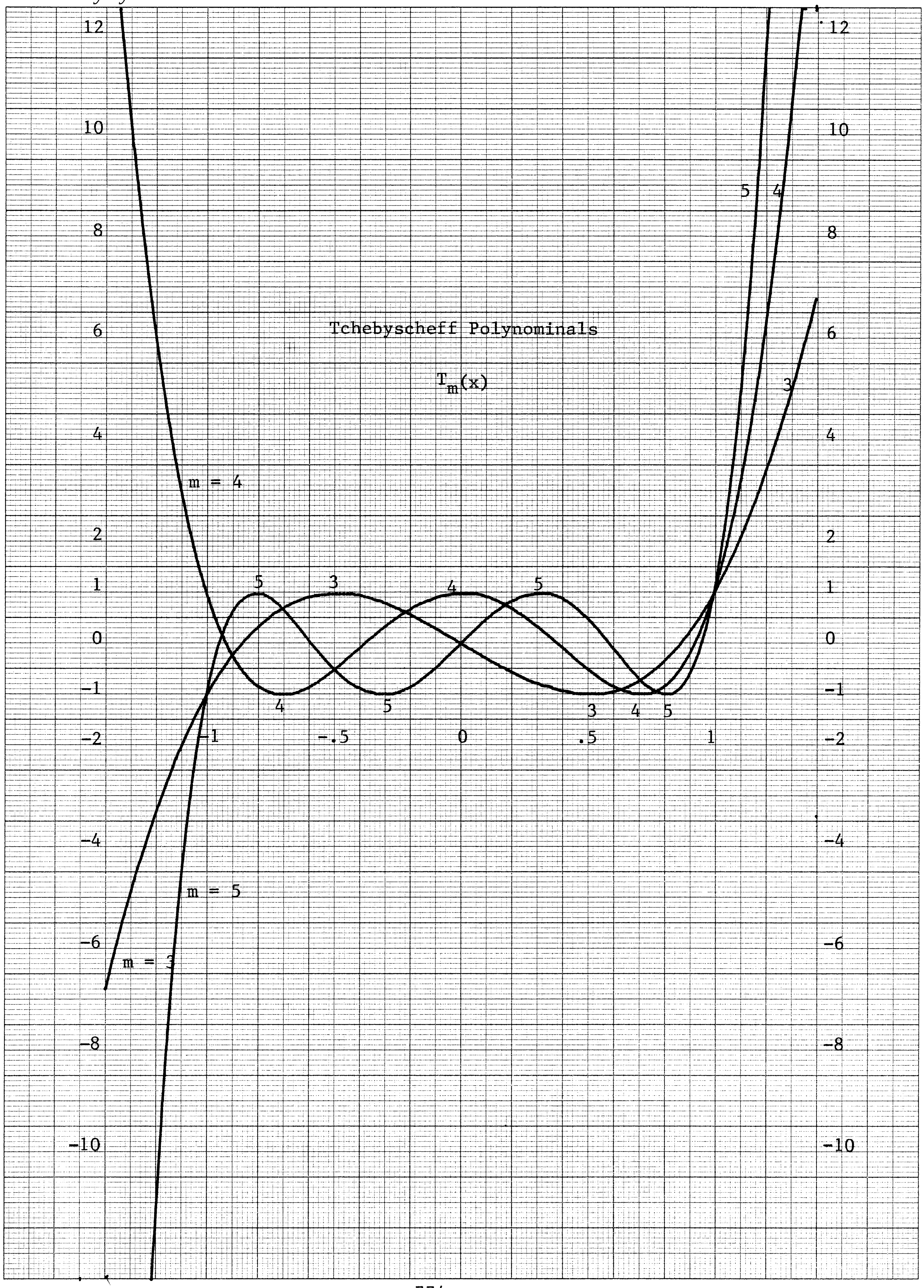
We can take the elements in pairs to get the array factor. If we use

$$\psi = k\,d \operatorname{Cos}\theta + \delta$$

then for one pair the array factor is

$$I_n(e^{j\,n\psi} + e^{-j\,n\psi}) \qquad \text{Odd}$$

$$I_n(e^{j\,(2n-1)\psi/2} + e^{-j\,(2n-1)\psi/2}) \qquad \text{Even}$$


Tchebyscheff Polynominals
T m (x)
m = 4
m = 5
m = 3
12
10
8
6
4
2
1
0
−1
−2
−4
−6
−8
−10
−1
−.5
0
.5
1
5
4
3

For odd number of elements, the total is given by 2N + 1 and for even 2N. We can combine the exponential functions into cosine and obtain the following array factors.

$$1 + 2 \sum_{n=1}^{N} \frac{I_n}{I_0} \text{Cos } 2n\,\psi/2 \qquad \text{Odd}$$

$$2 \sum_{n=1}^{N} \frac{I_n}{I_0} \text{Cos } (2n-1)\,\psi/2 \qquad \text{Even}$$

By using factors such as

$$\text{Cos }(2\psi/2) = 2\,\text{Cos}^2(\psi/2) - 1$$

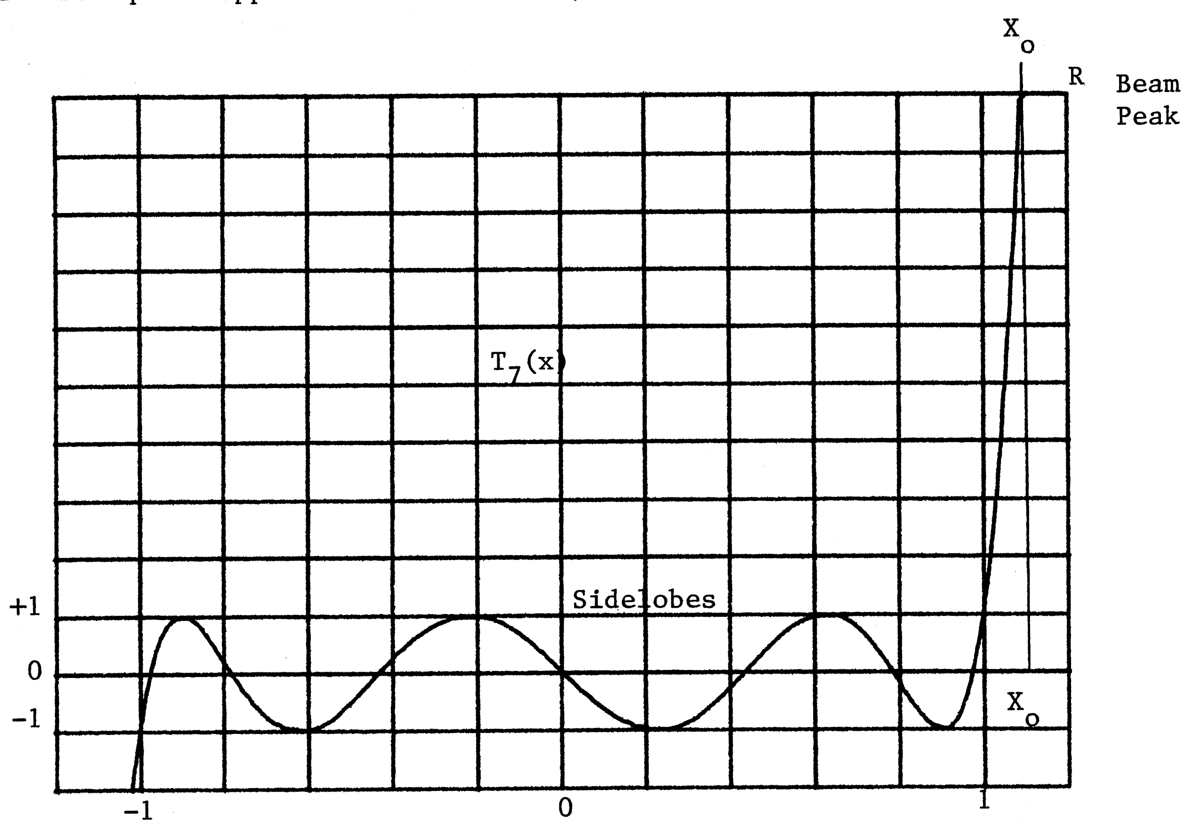
$$\text{Cos }(4\psi/2) = 8\,\text{Cos}^4(\psi/2) - 8\,\text{Cos}^2(\psi/2) + 1$$

$$\text{Cos }(3\psi/2) = 4\,\text{Cos}^3(\psi/2) - 3\,\text{Cos}(\psi/2)$$

$$\text{Cos }(5\psi/2) = 16\,\text{Cos}^5(\psi/2) - 20\,\text{Cos}^3(\psi/2) + 5\,\text{Cos}(\psi/2)$$

we can expand the above array factors as polynominals in the factor Cos($\psi/2$). We will find it necessary to expand the polynominal in terms of a constant, X_o, times Cos($\psi/2$) in order to relate the array factor to a Tchebyscheff polynominal and obtain the desired sidelobe level.

The peak of the beam occurs when $\psi = 0$. If we make this correspond to a value of X_o where the Tchebyscheff polynominal has a value R, then the sidelobes will be equal ripple and at a level 1/R.



If we make the substitution: $X = X_o \text{ Cos}(\psi/2)$, then we can use the Tchebyscheff polynominal with

$$T_m(X_o) = R$$

where 20 Log R is the desired sidelobe level for the array polynominal. The zeros of the Tchebyscheff polynominal are given by

$$X_p = \pm \text{ Cos}((2p - 1)\pi/(2m))$$

Using the equation

$$X_p = X_o \text{ Cos}(\psi/2) = X_o(e^{j\psi/2} + e^{-j\psi/2})$$

we can find the angles of the zeros in the W plane which are symmetrical.

$${}_p = \pm 2 \text{ Cos}^{-1}(X_p/X_o)$$

Once we have the zeros in the W plane, we can multiply out the roots form of the polynominal to find the coefficients of the array.

$$X_o = \text{Cosh}((\text{Cosh}^{-1}R)/m)$$

Example: Design a 10 element array with 25 dB sidelobes.

$m = 9$ $\quad R = 10^{(25/20)} = 17.7828$ $\quad X_o = 1.0797$

p	X_p	ψ_p
1	.9848	±48.41°
2	.8660	±73.34°
3	.6428	±106.93°
4	.3420	±143.06°
5	0	180°

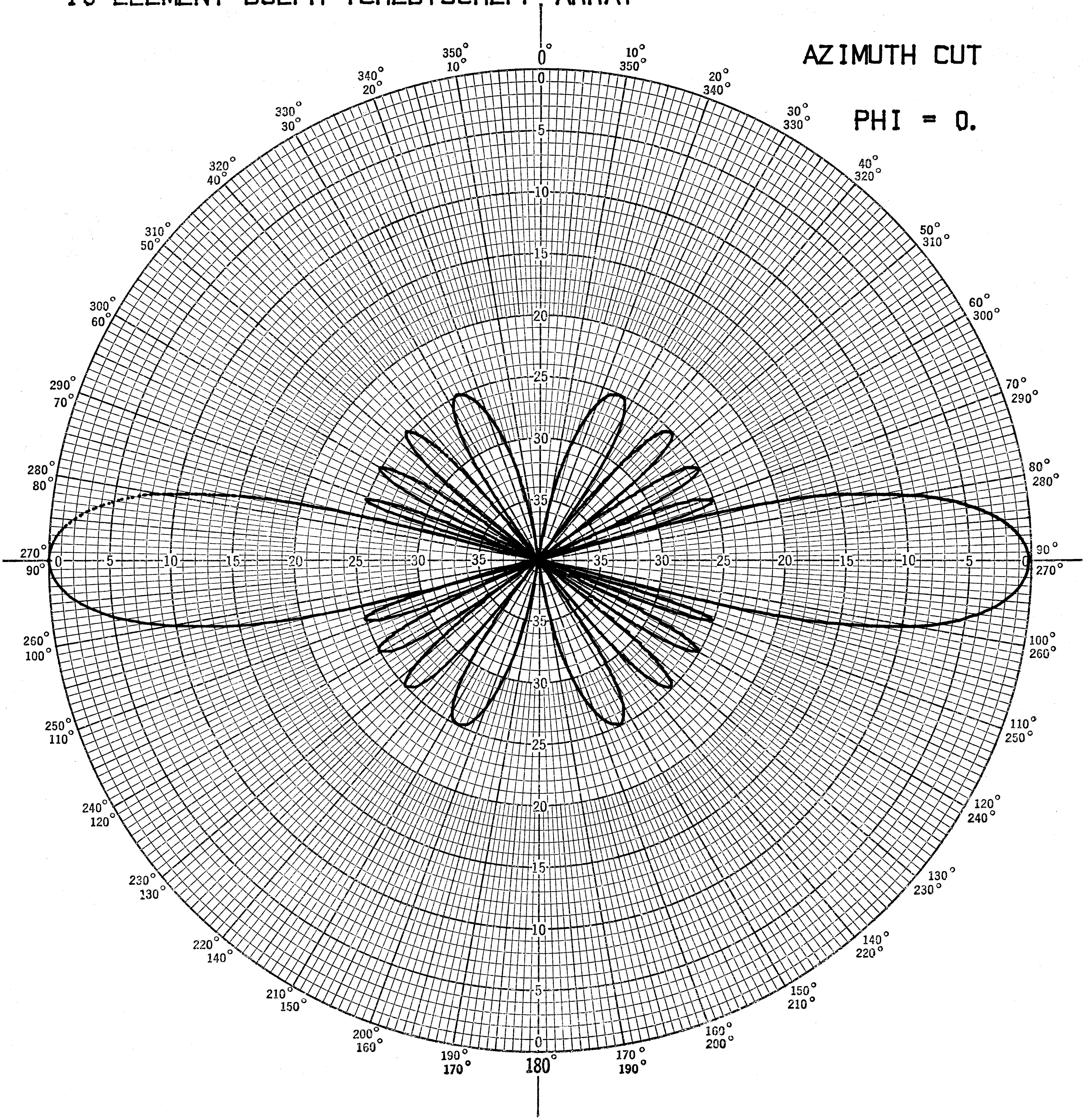
When we multiply out the root form of the polynominal and convert the voltage magnitudes of the coefficients to dB, we get the coefficients of the array.

NO.	Coefficient	NO.	Coefficient
1	-15.46 dB	10	-15.46
2	-13.31	9	-13.31
3	-10.23	8	-10.23
4	-8.31	7	-8.31
5	-7.39	6	-7.39

Since the zeros are in complex conjugate pairs, the phase angles of the elements are zero. The coefficients are symmetrical about the center of the array because the zeros are all on the unit circle.

The Schelkunoff W plane diagram is drawn on page 777 for the example for $\lambda/2$ spacings. A polar pattern is drawn on page 778 to more closely show the sidelobe structure.

10 ELEMENT DOLPH TCHEBYSCHEFF ARRAY



Polar Chart No. 127D
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OTHER TCHEBYSCHEFF TECHNIQUES

We can use the Tchebyscheff polynominals to arrive at other equal sidelobe designs. When the spacing between elements is less than $\lambda/2$, then the preceding method will not give an optimum design. We will need to restrict the designs to an odd number of elements. Start with an array factor for an odd number of elements symmetrically fed about the center line. From page 775 we have the following array factor

$$1 + \sum_{n=1}^{(N-1)/2} \cos n\psi$$

We will equate this to the Tchebyscheff polynominal by the transformation:

$$X = a \cos\psi + b$$

The visible reqion will range from X_o at the beam peak to $X = -1$ at the pattern minimum. In this case for N elements we equate the polynominal of the array to the $(N - 1)/2$ Tchebyscheff polynominal. Each root of the polynominal is used twice: once on the positive portion of the unit circle and once on the negative portion. These are in complex conjugate pairs so that the unscanned array has equal phase feed. If we substitute in the two restrictions, we can solve for the constants a and b. Beam peak occurs at $\psi = 0$.

$$X_o = a + b$$

The edges of the pattern occur at

$$\psi = -\kappa d + \delta$$

$$\psi = \kappa d + \delta$$

which we make correspond to $X = -1$. This depends on the value of δ

Suppose $\delta = 0$ and $\kappa d < \pi$ (less than $\lambda/2$ spacings), then

$$-1 = a \cos \kappa d + b$$

We can solve for a and b.

$$a = \frac{1 + X_o}{1 - \cos \kappa d} \qquad b = \frac{-(1 + X_o \cos \kappa d)}{1 - \cos \kappa d}$$

Suppose we want an array with 9 elements at a spacing of 0.375 with 25 dB sidelobes.

$$(N - 1)/2 = 4$$

We use a fourth order Tchebyscheff polynominal to find the roots.

$$X_o = \text{Cosh}(\,\text{Cosh}^{-1}(10^{25/20})/4) = 1.4256$$

$$a = \frac{1 + 1.4256}{1 - \text{Cos}(135°)} = 1.4209$$

$$b = \frac{-(1 + 1.4256\ \text{Cos}(135°)}{1 - \text{Cos}(135°)} = 0.0047$$

Using these coefficients and the transformation

$$X = a\ \text{Cos}\ \psi + b$$

we can relate the zeros of $T_4(x)$ to the zeros on the unit circle.

$$\psi_p = \pm\ \text{Cos}^{-1}\left(\frac{X_p - b}{a}\right)$$

$$X_p = \pm\ \text{Cos}((2p - 1)\pi/2m)$$

	X_p	ψ_p	ψ_p
1	±0.9239	±49.69°	±130.81°
2	±0.3827	±74.57°	±105.82°

When we place these 8 zeros on the unit circle, we can find the pattern and coefficients of the array. These are given on page 781. If we have an even number of elements, then we must place one of the zeros at -1 to maintain an equal phase feed. This will not give us equal sidelobes if we use the design for an odd number of elements and add the zero at -1 in the W plane. The pattern of a 10 element array using this technique is shown on page 782 and we see that the sidelobes are not equal.

If kd is greater than π, then the pattern will extend further than once around the unit circle in the W plane and we use the following constants.

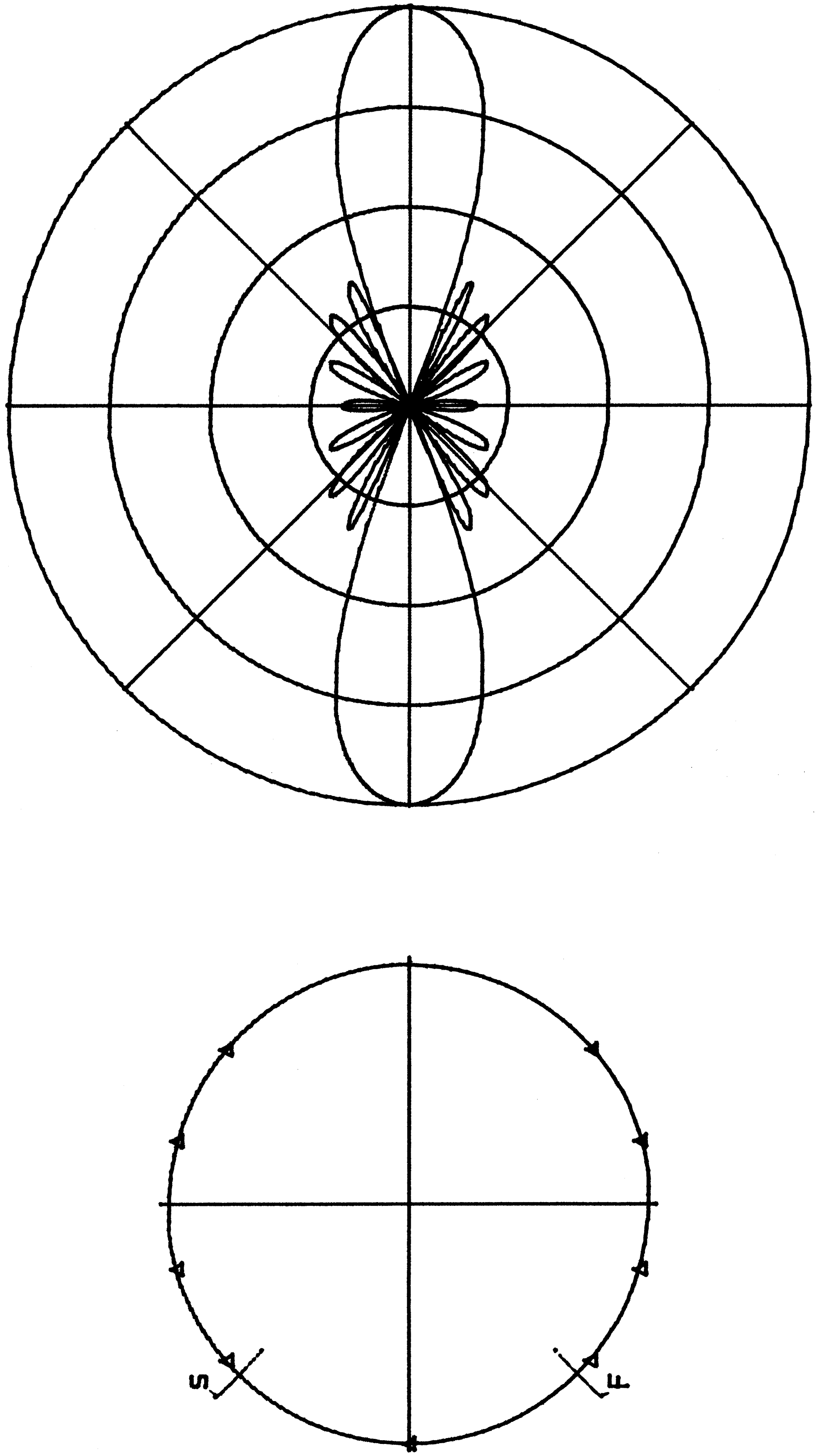
$$a = \frac{1 + X_o}{2} \qquad b = \frac{X_o - 1}{2} \qquad kd \geq \pi$$

For an odd number of elements, this technique will give the same zeros as the former technique which is good for both odd and even number of elements when the spacing between elements is greater than or equal to a half wavelength.

Suppose $\delta \neq 0$, then the beam will be scanned. We must adjust the zeros in the W plane to still give and equal sidelobe response. The visible region varies over the range:

$$-\ kd + \delta \quad \text{to} \quad kd + \delta$$

We divide this into two cases. The first one does not include the point -1 of the W plane in the visible region. Then we have similiar equations for a and b.



10 ELEMENT ARRAY. 0.375 SPACINGS

$$a = \frac{1 + X_o}{1 - \mathrm{Cos}(k\ d - \delta)} \qquad b = \frac{-(1 + X_o \mathrm{Cos}(kd - \delta)}{1 - \mathrm{Cos}(k\ d - \delta)}$$

Example: Design a 9 element array with 0.25 spacings scanned to 45°. As before we require 25 dB sidelobes.

In order to scan to 45° the phase shift between elements, δ , is - kd Cos θ_m

$$\delta = -63.64°$$

$$X_o = 1.4256 \quad \text{with} \quad m = 4$$

$$a = \frac{1 + 1.4256}{1 - \mathrm{Cos}(153.64°)} = 1.2793$$

$$b = \frac{-(1 + 1.4256\ \mathrm{Cos}(153.64°)}{1 - \mathrm{Cos}(153.64°)} = 0.1463$$

We use the transformation: $X = a\ \mathrm{Cos}\ \psi + b$

$$\psi_p = \pm \mathrm{Cos}^{-1}((X_p - b)/a) \qquad X_p = \pm \mathrm{Cos}((2p - 1)\pi/(2m))$$

X_p	ψ_p	ψ_p
±0.9239	±52.57°	±146.77°
±0.3827	±79.35°	±114.42°

These zeros are placed on the unit circle, and the pattern and coefficients of the array are given on the top of page 784. The beam is scanned to 45° and the sidelobes are 25 dB as specified. Four of the zeros are out of visible space means that there is not an excessive amount of stored energy about the array and that the efficiency will be good. Also the values of the feeding coefficients are not critical to achieve the sidelobe level.

In the second case of a scanned array the point -1 in the W plane is included in the visible region for a scanned beam. For this case we use the formulas on page 780 for a and b.

$$a = \frac{1 + X_o}{2} \qquad b = \frac{X_o - 1}{2}$$

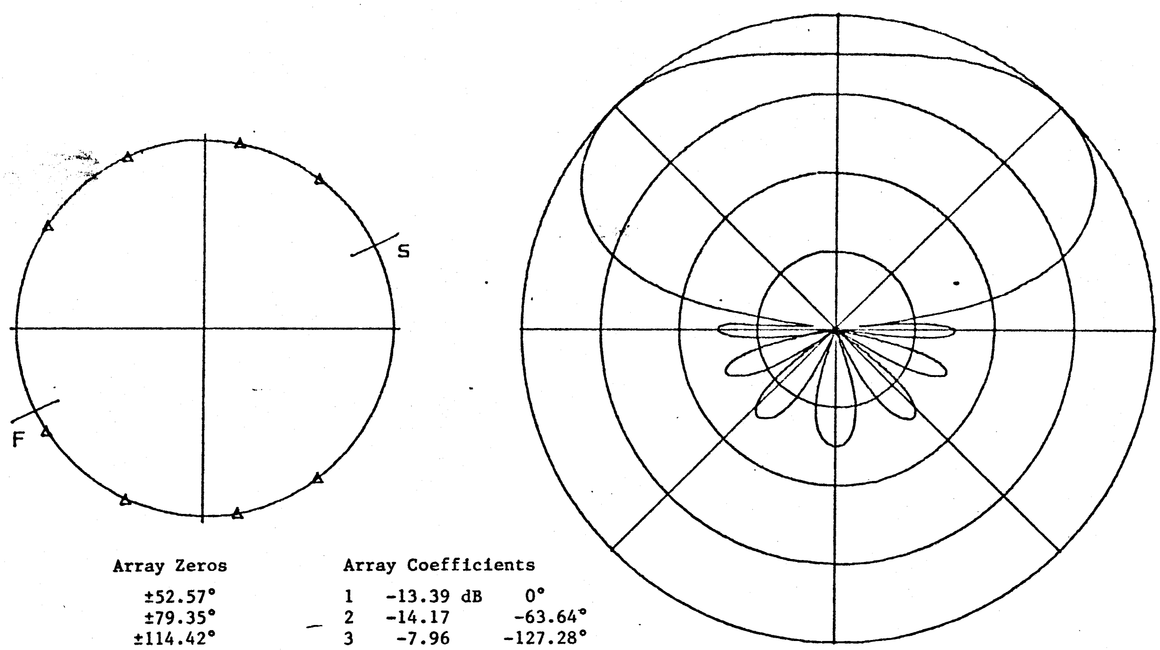
Example: Design a 9 element array with 0.375λ spacings scanned to 45° with 25 dB sidelobes.

This is almost identical to the example above only the point -1 in the W plane is included in the visible region.

$$X_o = 1.4256 \qquad a = 1.2128 \qquad b = 0.2128$$

The phase shift between elements is given by: $\delta = -k\ d\ \mathrm{Cos}\ \theta_m = -95.46°$
The zeros are found by using the transformation between W plane zeros and the Tchebyscheff polynominal.

9 ELEMENTS 0. 25 SPACINGS, −63. 64 DEG BETWEEN



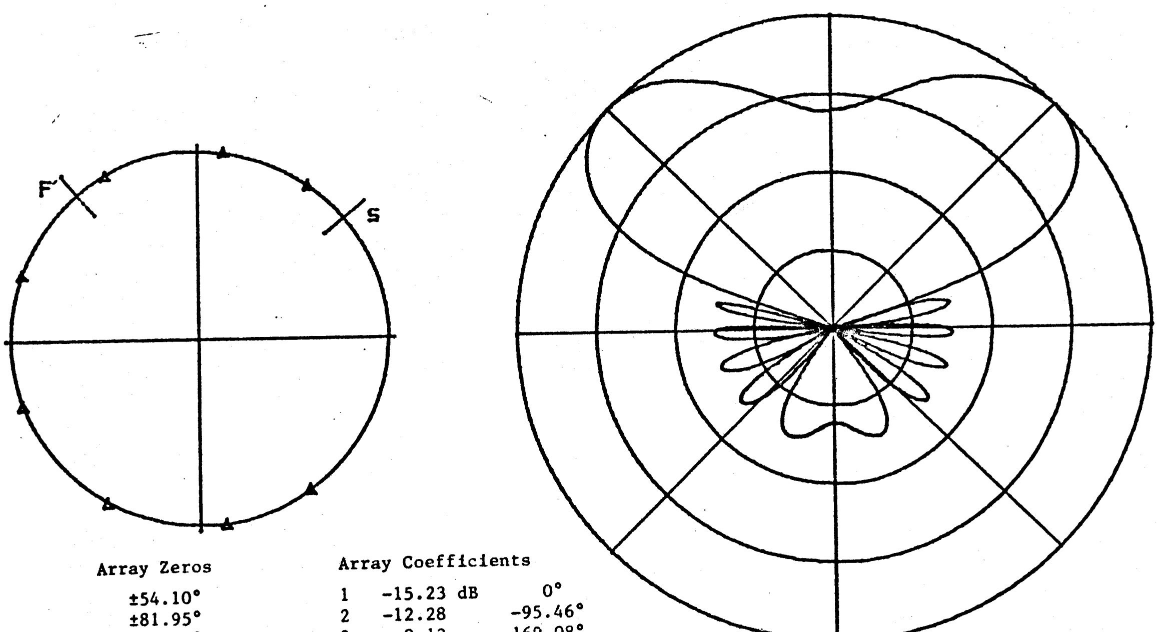
Array Zeros

±52.57°
±79.35°
±114.42°
±146.77°

Array Coefficients

1	−13.39 dB	0°
2	−14.17	−63.64°
3	−7.96	−127.28°
4	−8.92	169.08°
5	−5.93	105.44°
6	−8.92	41.80°
7	−7.96	−21.84°
8	−14.17	−85.48°
9	−13.39	−149.12°

9 ELEMENT ARRAY 0. 375 SPACINGS, −95. 46 DEG



Array Zeros

±54.10°
±81.95°
±119.41°
±159.59°

Array Coefficients

1	−15.23 dB	0°
2	−12.28	−95.46°
3	−9.13	169.08°
4	−7.36	73.62°
5	−6.79	−21.84°
6	−7.36	−117.30°
7	−9.13	147.24°
8	−12.28	51.78°
9	−15.23	−43.68°

X_p	p	p
±0.9239	±69.17°	±159.59°
±0.3827	±81.95°	±119.40°

The W plane diagram and pattern are drawn on the bottom of page 784. The beam is scanned to 45° and the sidelobes are 25 dB maximum. Because the total array is longer, the beamwidth is smaller than the $\lambda/4$ spaced array. One more W plane zero is included in the visible region and shows as an extra null in the pattern.

If we use this method for arrays scanned to endfire, then we obtain a design with equal ripple, but it is not optimum. An optimum design is obtained by letting the beam maximum correspond to $-X_o$. The phase shift between elements is made greater than necessary for endfire.

$$-X_o = a \operatorname{Cos}(k\,d + \delta) + b$$

The pattern at $\psi = 0$ is at a maximum of the ripple, X = 1.

$$a + b = 1$$

The other end of the visible region corresponds to X = -1

$$-1 = a \operatorname{Cos}(k\,d - \delta) + b$$

We can solve for the three unknowns:

$$a = \frac{X_o + 3 + 2 \operatorname{Cos}(kd) \sqrt{2X_o + 2}}{2 \operatorname{Sin}^2(kd)}$$

$$b = 1 - a$$

$$\delta = \operatorname{Sin}^{-1}((X_o - 1)/(2\,a\,\operatorname{Sin}(kd)))$$

Example: Design a 9 element endfire array with 0.25λ spacings and 25 dB sidelobes.

$$X_o = 1.4256 \quad \text{with} \quad m = 4$$

$$a = 2.2128 \qquad b = -1.2128 \qquad \delta = 5.52°$$

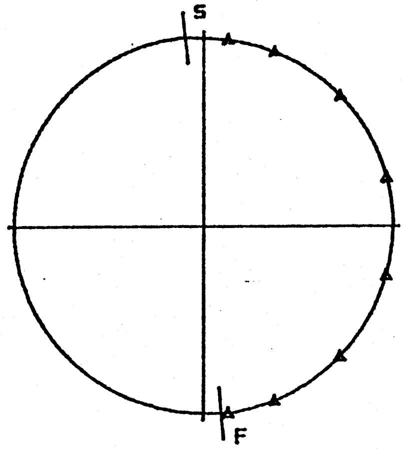
The zeros in the W plane are found in the usual manner.

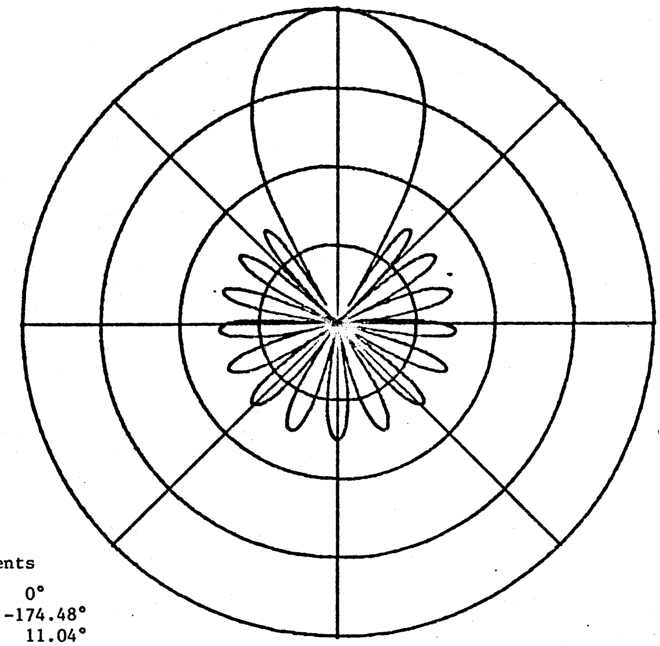
$$X_p = \pm \operatorname{Cos}((2p - 1)\pi/(2m)) \qquad \psi_p = \pm \operatorname{Cos}^{-1}((X_p - b)/a)$$

ψ_p: ±15.07°, ±43.86°, ±67.97°, ±82.50°

The array pattern is plotted on the top of page 786. On the bottom is the pattern of an array designed by the former synthesis procedure. There is quite an improvement in the directivity with this method. When we look at the

9 ELEMENTS 0.25 SPACINGS 25 DB SIDELOBES ENDFIRE





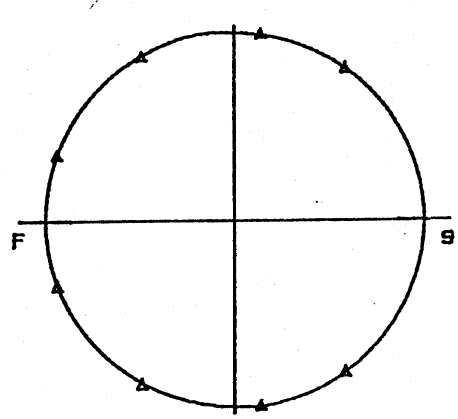
Angles of Zeros

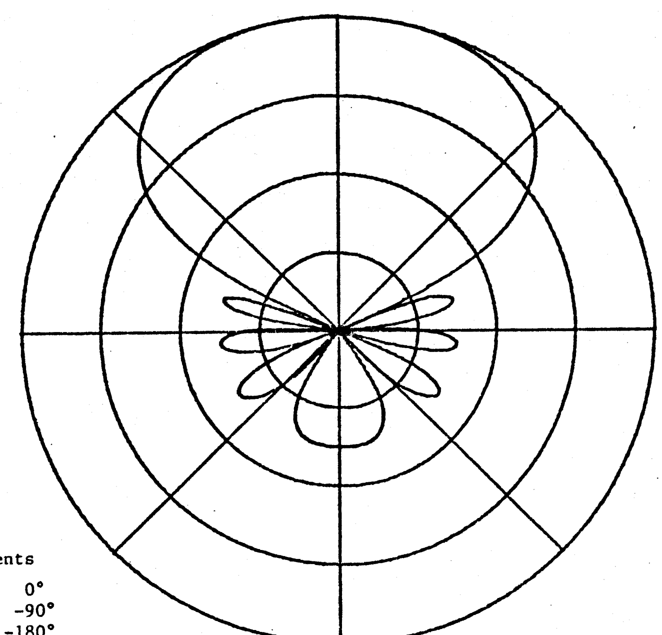
±15.07°
±43.86°
±67.97°
±82.50°

Array Coefficients

1	-30.73 dB	0°
2	-17.89	-174.48°
3	-10.40	11.04°
4	-6.31	-163.45°
5	-5.01	22.07°
6	-6.31	-152.41°
7	-10.40	33.11°
8	-17.89	-141.37°
9	-30.73	44.15°

9 ELEMENT ARRAY 0.25 SPACINGS ENDFIRE





Angles of Zeros

±54.10°
±81.95°
±119.41°
±159.59°

Array Coefficients

1	-15.23 dB	0°
2	-12.28	-90°
3	-9.13	-180°
4	-7.36	90°
5	-6.79	0°
6	-7.36	-90°
7	-9.13	-180°
8	-12.28	90°
9	-15.23	0°

phase angles between elements, we see that they are nearly 180°. In an array this would mean that the input impedance would be quite low and the efficiency quite low because of the high value of stored energy. The array on the bottom has lower directivity, but the efficiency will be higher.

ZERO SAMPLING OF CONTINUOUS DISTRIBUTIONS

An array can be designed by sampling an aperture distribution such as a Taylor line source. When the array is large, the sampling will accurately represent the distribution. This method avoids the necessity of multiplying out long polynominals which can have numerical difficulties. Usually double precision must be used in the computer program to reduce the accumulated round off and truncation errors. When the array is small, sampling will not give the same response as the aperture. We can improve the match by sampling the nulls of the aperture distribution pattern. If we consider the k d Cos θ space pattern of an array as drawn on page 106 and 107, we can see that the pattern of an array repeats at 2π intervals. The aperture distribution k Cos θ pattern has no repeat interval.

When the spacings between the elements of an array are $\lambda/2$, then the visible region in k d Cos θ space spans 2π or the array repeat interval. This means that we must equate an array with $\lambda/2$ spacings to an aperture with the same length regardless of the actual spacings between elements. The effective length of an array sampling a distribution is one element longer than the actual array because we consider each element to be centered on the sampling interval of the array. The first and last elements then each extend a half element spacing beyond the array. For uniform amplitudes the zeros of the aperture will match the zeros of the array. The zeros of the uniform distribution

$$\frac{\sin \pi U}{\pi U}$$

are modified for a Taylor distribution. The unmodified zeros of the uniform distribution occur at integer values of U. A k space pattern in the variable U has been drawn on page 788 for a Taylor line source with the first unmodified zero at U = 4 and a sidelobe limit of 25 dB. For a uniform amplitude array the zeros in the W plane are given by

$$W_i = e^{j2\pi i/N} \qquad i = 1, 2, \ldots, N-1$$

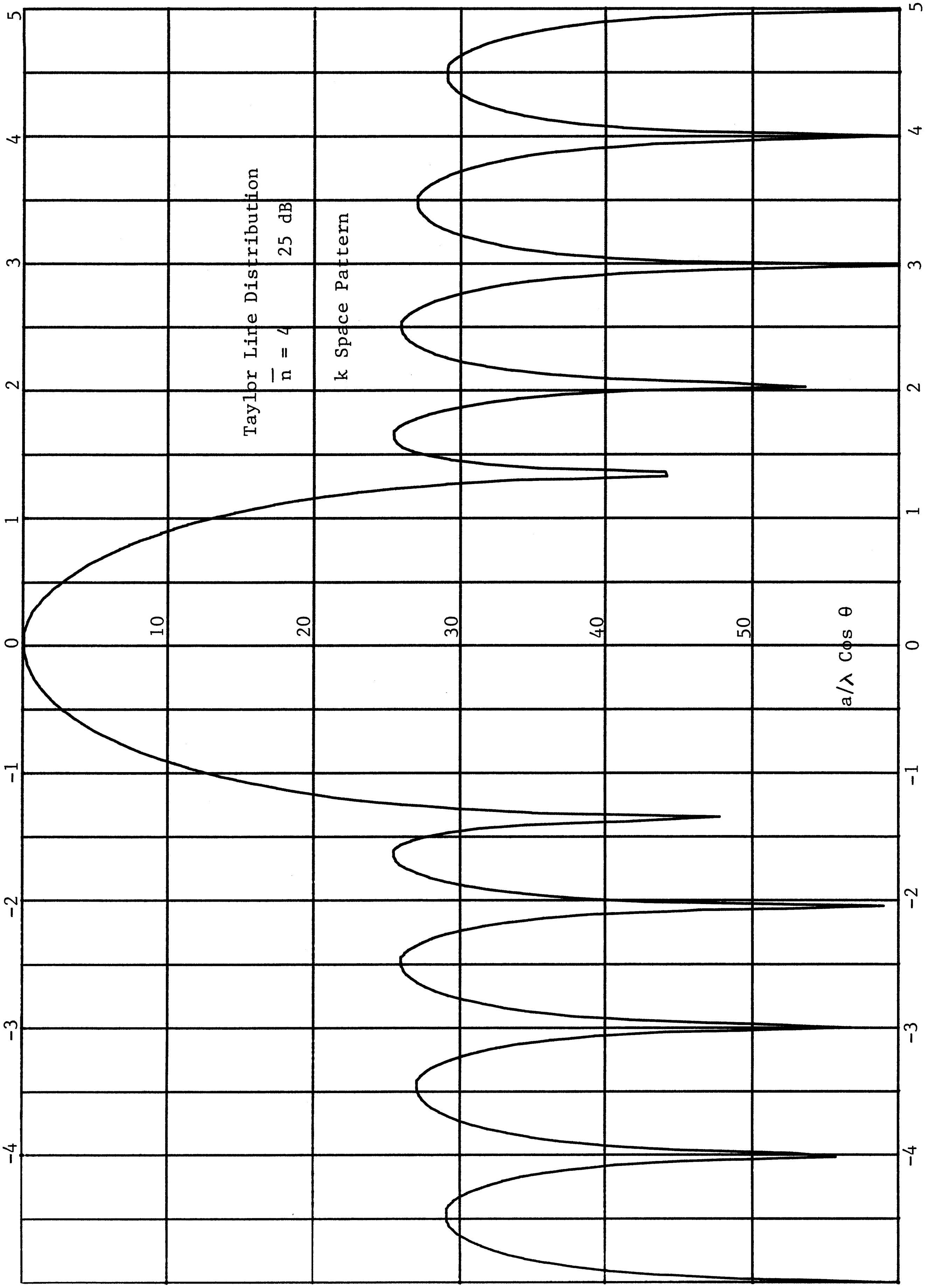
where i are integers. If the modified zeros of the Taylor are given by U_i, then the zeros in the W plane are given by

$$W_i = e^{j2\pi U_i/N}$$

for N elements in the array.

Suppose we zero sample the Taylor distribution pattern drawn on page 788. The zeros of the distribution pattern are

±1.3497,	± 2.0457,	±2.9851,	±4,	5	
±48.59°	±73.64°	±107.46°	±144°	180°	W Plane Zeros


Taylor Line Distribution
n̄ = 4   25 dB
k Space Pattern
a/λ Cos θ
10
20
30
40
50
5
4
3
2
1
0
−1
−2
−3
−4

These W plane zeros which are all on the unit circle were used to design a 10 element array with 0.75 wavelength spacings between elements. The pattern is drawn on page 790. The sidelobes are maximum 25 dB as the uniform distribution although the outer sidelobes do not fall off at the same rate as the aperture distribution. We can also design the array by sampling the distribution. On page 791 is a plot of the Taylor distribution with the indicated sampling points. Another array was designed using these values and is drawn on page 792. In this case the sidelobes are greater than 25 dB which means that sampling the distribution will not give as good a pattern response as sampling the zeros of the k space distribution of the aperture. Below is a comparison of the two designs amplitudes.

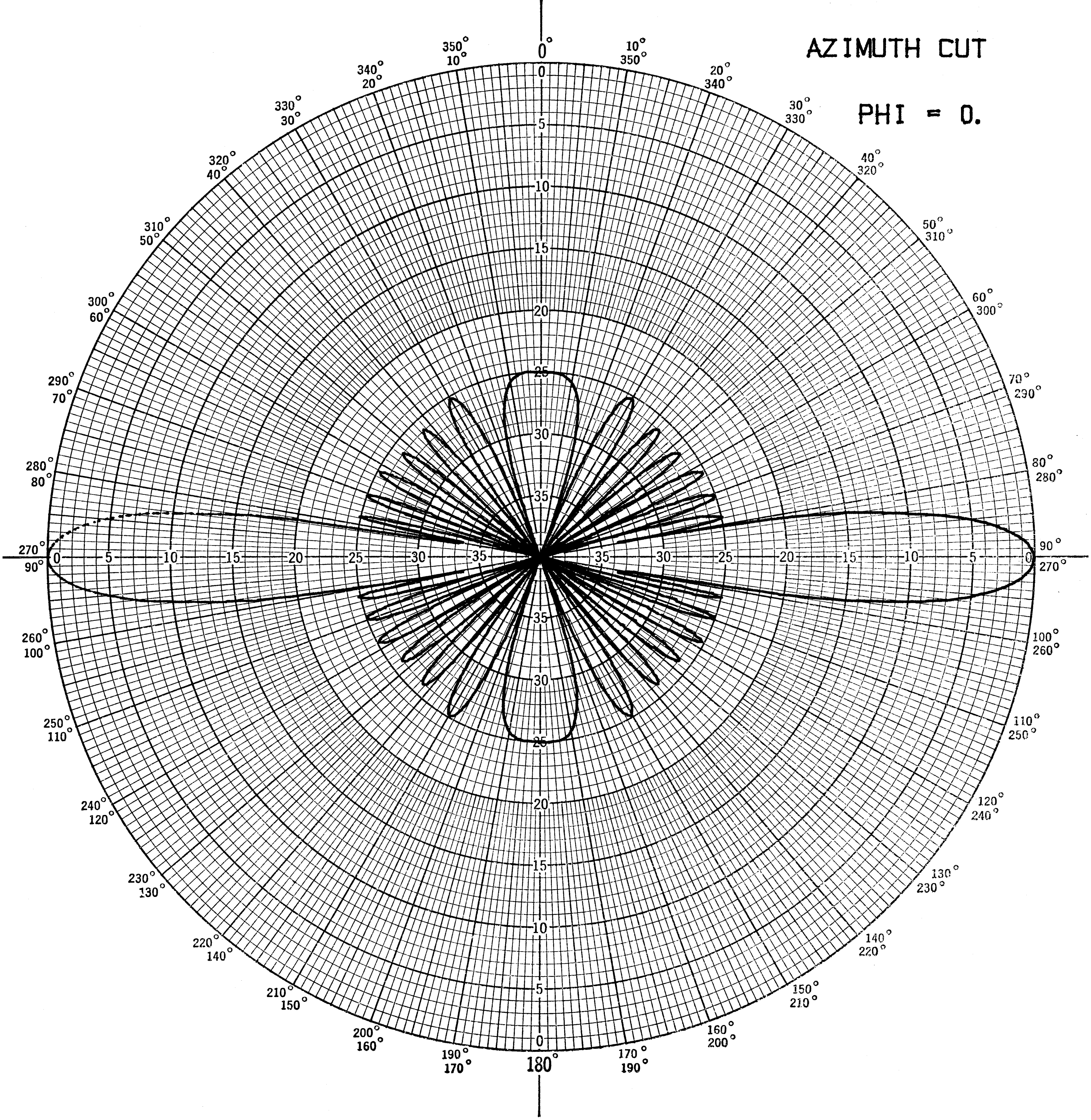
Element	Zero Sampled	Distr. Sampled
1	-8.32 dB	-8.17 dB
2	-5.83	-5.74
3	-2.91	-2.89
4	-0.90	-0.99
5	0	0
6	0	0
7	-0.90	-0.99
8	-2.91	-2.89
9	-5.83	-5.74
10	-8.32	-8.17

The differences between the designs are small but the zero sampled design more closely follows the continuous distribution. This method also works for any distribution such as variable sidelobe designs. In many of these methods it is the zeros of the pattern which are manipulated and not the distribution itself.

SHAPED BEAM METHODS

The previous methods seek to obtain the narrowest beamwidth for given sidelobes. We will now discuss methods of obtaining shaped beams with arrays. The two most used methods are Fourier series and Woodward-Lawson methods. Remember that any linear array must be circularly symmetric about the axis. We found the beamwidth of various linear apertures which can be related to linear arrays of uniformly spaced elements. On page 787 we consider the first and last elements to be extended a half element spacing beyond these elements when sampling an aperture distribution. We will use this length when comparing to aperture distributions. The effective length of an array is given by Nd where d is the interelement spacing and N is the number of elements. The physical length is $(N - 1)d$. We can use the results of Chapter 13 to find beamwidths. The usual result is for an unscanned broadside beam. To find the beamwidth of the scanned beam we can use the plot on page 555. If we need an array for a shaped beam, then it must be larger than required for the beamwidth to have degrees of freedom for the shaping. The beam cannot be shaped when the whole array size is required to establish the beamwidth. The array can more accurately shape the pattern when it is larger.

10 ELEMENTS ZERO SAMPLED TAYLOR DISTR. N = 4. 25 DB



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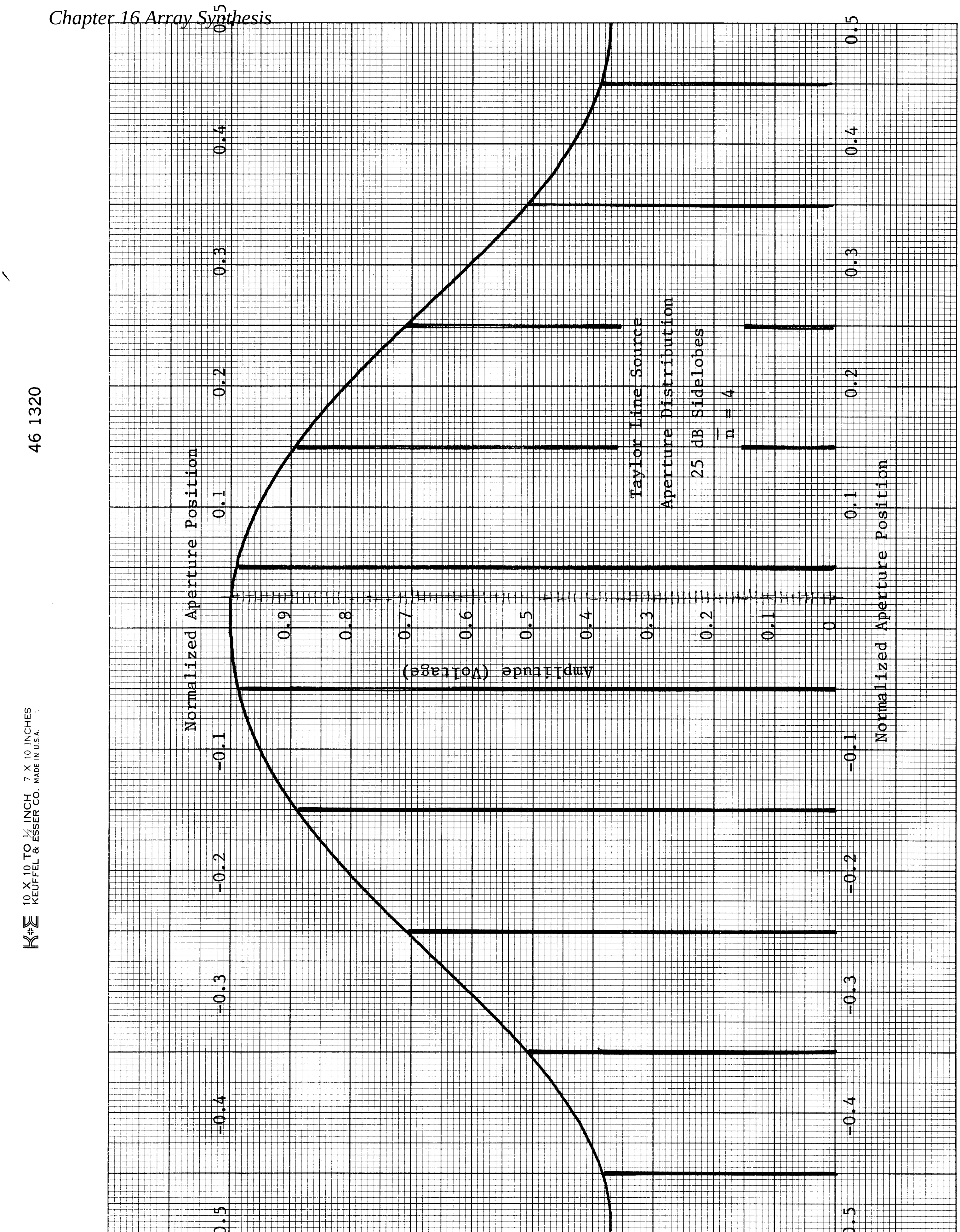
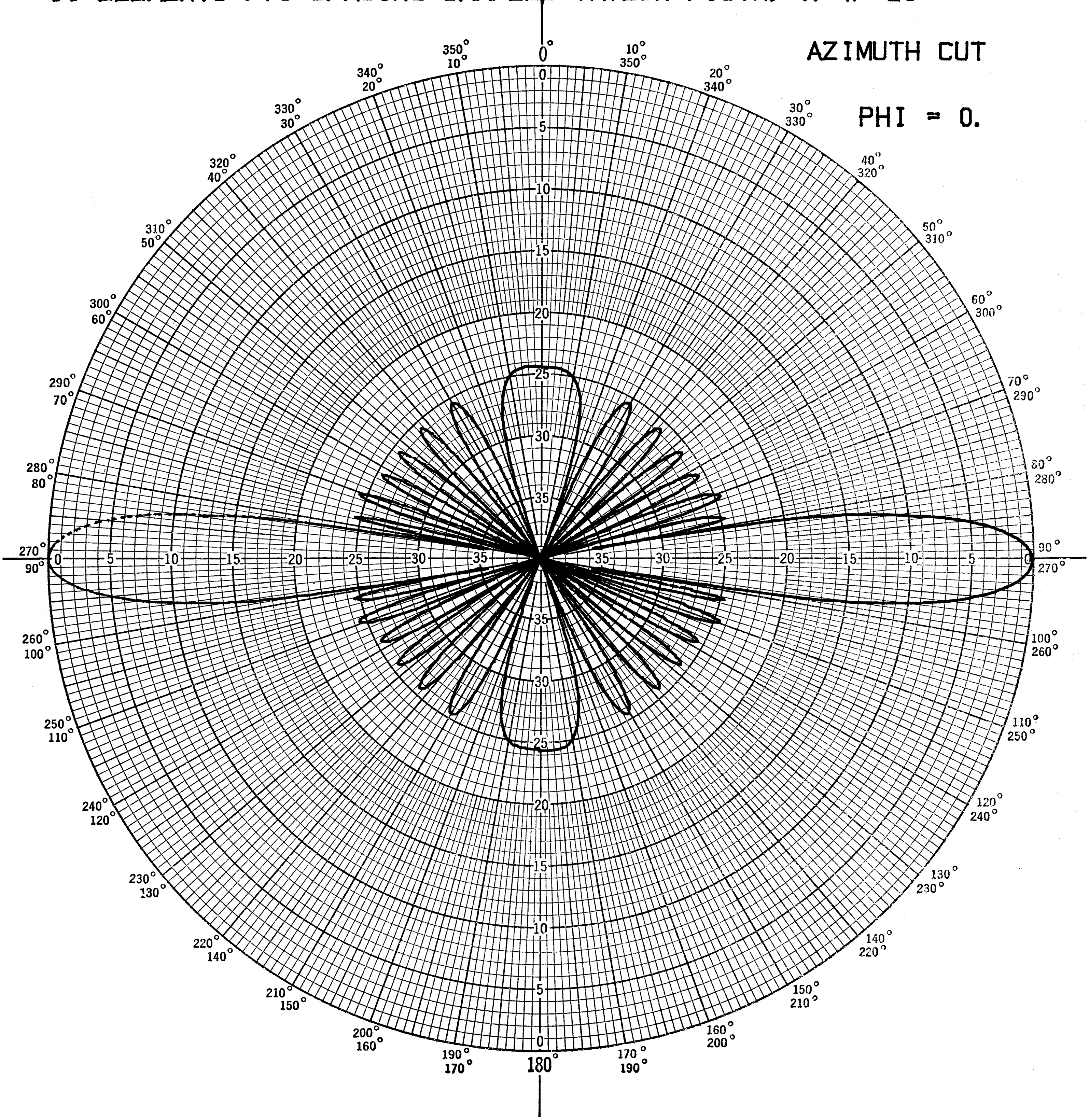

Taylor Line Source
Aperture Distribution
25 dB Sidelobes
$\bar{n} = 4$
Normalized Aperture Position
Amplitude (Voltage)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.5
0.4
0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.4
-0.5
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

10 ELEMENTS .75 SPACING SAMPLED TAYLOR DISTR. N=4, 25



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FOURIER SERIES METHOD

We discussed the Fourier transform method of apertures on page 530. The k space pattern of an aperture is defined from $-\infty$ to $+\infty$. The array pattern is periodic in k space which implies that Fourier series are used to expand the pattern. The array pattern for a symmetrically fed array is given by (pp. 775)

$$1 + 2 \sum_{n=1}^{m} \frac{I_n}{I_o} \operatorname{Cos}(2n\psi/2) \qquad \text{Odd}$$

or

$$2 \sum_{n=1}^{m} \frac{I_n}{I_o} \operatorname{Cos}(2n-1)\psi/2 \qquad \text{Even}$$

where $m = 2N + 1$ (odd) or $m = 2N$ (even) with $\psi = k\, d \operatorname{Cos}\theta + \delta$. These expressions show that the higher harmonics of the pattern are established by the further out elements. We will not necessarily have symmetric array coefficients. The array pattern is expressed as

$$f(\psi) = \sum_{n=-m}^{m} a_n e^{jn\psi} \qquad \text{Odd}$$

$$f(\psi) = \sum_{n=1}^{m} (a_n e^{j(2n-1)\psi/2} + a_{-n} e^{-j(2n-1)\psi/2}) \qquad \text{Even}$$

Suppose $f_d(\psi)$ is the desired pattern in k space. We can expand this in a Fourier series since the period is $\pm\pi$ in ψ space.

$$f_d(\psi) = \sum_{n=-\infty}^{\infty} a_n e^{jn\psi} \qquad \text{Odd}$$

$$f_d(\psi) = \sum_{n=1}^{\infty} (a_n e^{j(2n-1)\psi/2} + a_{-n} e^{-j(2n-1)\psi/2}) \qquad \text{Even}$$

We can equate the coefficients of the two Fourier series. The array is the first m coefficients of the expansion.

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_d(\psi)\, e^{-jn\psi}\, d\psi \qquad \text{Odd}$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_d(\psi)\, e^{-j(2n-1)\psi/2}\, d\psi$$

Even

$$a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_d(\psi)\, e^{j(2n-1)\psi/2}\, d\psi$$

The array coefficients are found directly from the Fourier series coefficients.

Example. Design a 21 element array with $\lambda/2$ spacings with a constant beam 2b wide in ψ space.

$$a_n = \frac{1}{2\pi}\int_{-b}^{b} e^{-jn\psi}\, d\psi = \frac{\mathrm{Sin}\ nb}{\pi n}$$

Suppose the constant beamwidth is 45° at broadside. $67.5° \le \theta \le 112.5°$.

$$b = \frac{360°}{\lambda}\frac{\lambda}{2}\,\mathrm{Cos}(67.5°) = 68.88°$$

We can ignore the constant factor $1/\pi$ and expand the expression to find the array coefficients.

n	a_n	Amplitude	Phase
0	1	0 dB	0°
±1	0.9328	-0.60	0°
±2	0.3361	-9.47	0°
±3	-0.1495	-16.50	180°
±4	-0.2488	-12.08	180°
±5	-0.0537	-25.40	180°
±6	0.1336	-17.48	0°
±7	0.1209	-18.35	0°
±8	-0.0240	-32.40	180°
±9	-0.1094	-19.22	180°
±10	-0.0518	-25.72	180°

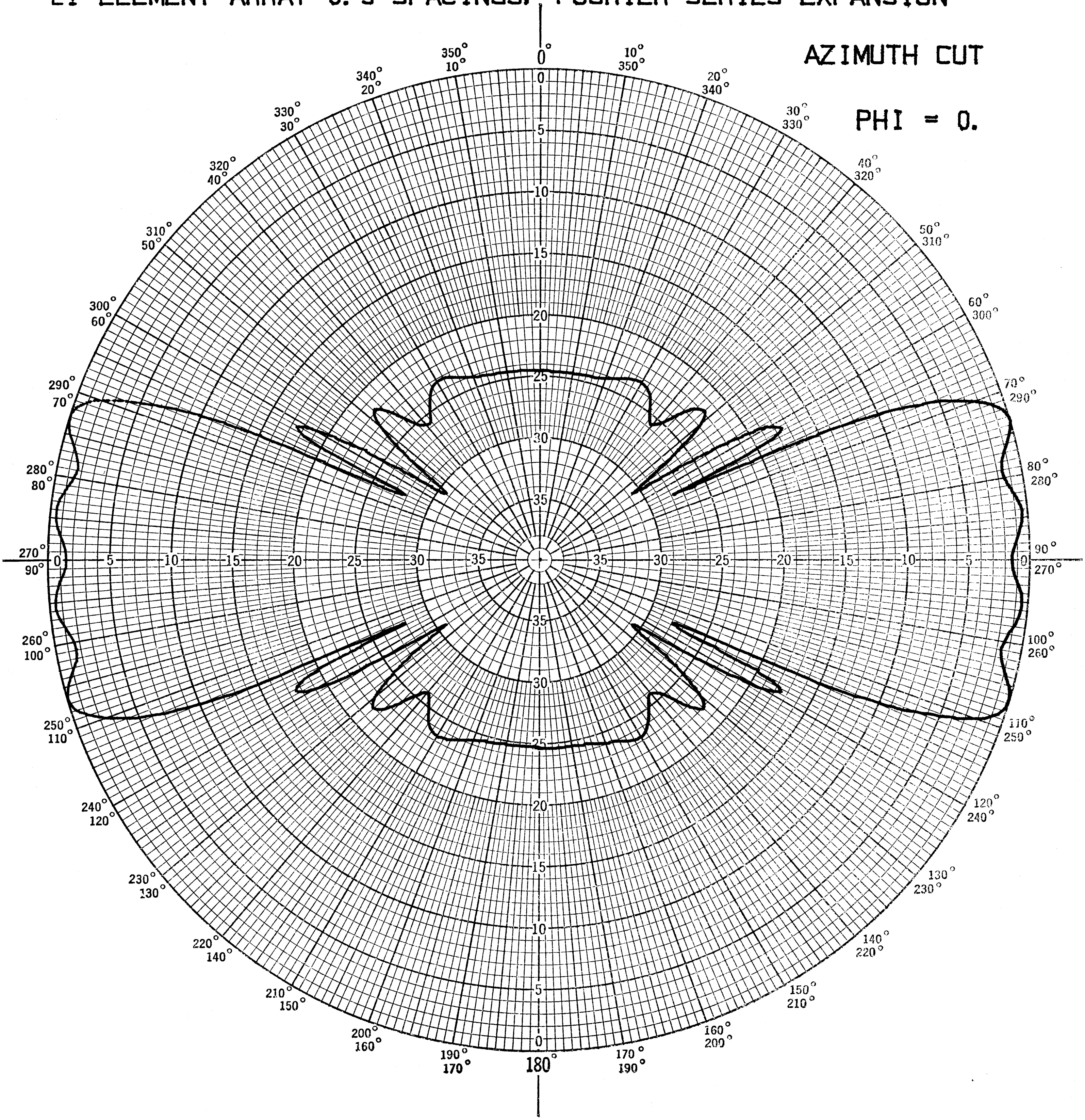
The array response is plotted on page 795. A Fourier series expansion gives an approximation which is least square for the given number of elements. As the array is made longer and longer, it would more closely approximate the desired pattern.

The example can be repeated with 20 elements. We use the integrals at the top of the page to find the coefficients.

$$a_n = \frac{1}{2\pi}\int_{-b}^{b} e^{-j(2n-1)\psi/2}\, d\psi = \frac{2\ \mathrm{Sin}((2n-1)b/2)}{(2n-1)}$$

We get the same result for a_{-n}. When we expand this, we get the following coefficients:

21 ELEMENT ARRAY 0.5 SPACINGS. FOURIER SERIES EXPANSION



Polar Chart No. 127D
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n	$a_{\pm n}$	Amplitude	Phase
±1	1	0 dB	0°
±2	0.5735	-4.83	0°
±3	0.0479	-26.38	0°
±4	-0.2211	-13.11	180°
±5	-0.1506	-16.45	180°
±6	0.0520	-25.69	0°
±7	0.1359	-17.34	0°
±8	0.0468	-26.60	0°
±9	-0.0742	-22.59	180°
±10	-0.0848	-21.44	180°

The response of this array is plotted on page 797. The pattern has a null at 0° and 180° because the array is symmetrical about the center and spaced $\lambda/2$. This is a 20 element version of the two element even mode array discussed on page 85 which also has nulls at $\theta = 0°$ and $\theta = 180°$. There is an odd number of half wavelengths between equal amplitude elements causing cancellation along the axis. It appears that this array is better than the one with 21 elements, but in the main beam area they are nearly identical.

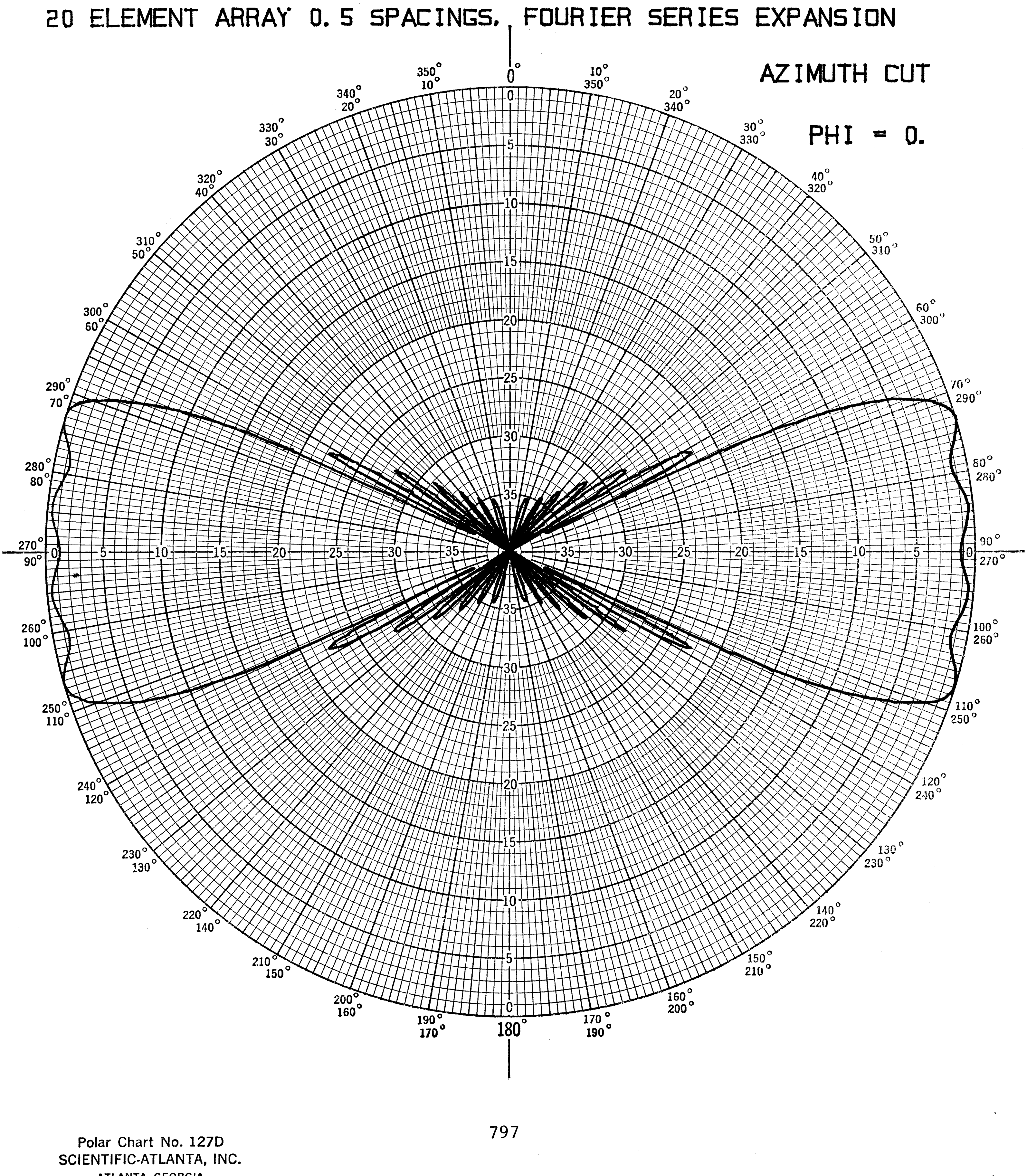
We can use this to design arrays with spacings greater than $\lambda/2$, but we can expect some problems because the Fourier series interval does not cover the total visible region. Take the above example and design an array with 0.75 spacings. To get an array about the same length, we use 13 elements.

$$b = \frac{360°}{\lambda} 0.75\lambda \, \text{Cos}(67.5°) = 103.32°$$

We use this constant with the expression for a broadside array to find the coefficients.

n	a_n	Amplitude	Phase
0	1	0 dB	0°
±1	0.9731	-0.24	0
±2	-0.2243	-12.98	180°
±3	-0.2554	-11.85	180°
±4	0.2004	-13.96	0°
±5	0.0794	-22.00	0°
±6	-0.1641	-15.70	180°

The pattern of this design is plotted on page 798. The procedure has failed to produce a usable design. The main beam is identical to the one produced by the half wavelength spaced array with 21 elements, but not enough of the visible region is covered by the Fourier integral to suppress the lobes beyond the main beam. If we design a series of arrays with spacings between 0.5λ and 0.75λ, we find that the array degrades smoothly from one to the other. The Fourier series method can only be used with arrays only slightly above 0.5λ spacings with good results. It also works well for arrays with spacings less than $\lambda/2$. An array designed with 0.375 spacings (27 elements) is down 30 dB along the axis which is better that the $\lambda/2$ spaced array. When more and more elements are used to sample the Sin X/ X aperture distribution for uniform pattern over a region of θ, then more of the higher


20 ELEMENT ARRAY 0.5 SPACINGS, FOURIER SERIES EXPANSION
AZIMUTH CUT
PHI = 0.
Polar Chart No. 127D
SCIENTIFIC-ATLANTA, INC.
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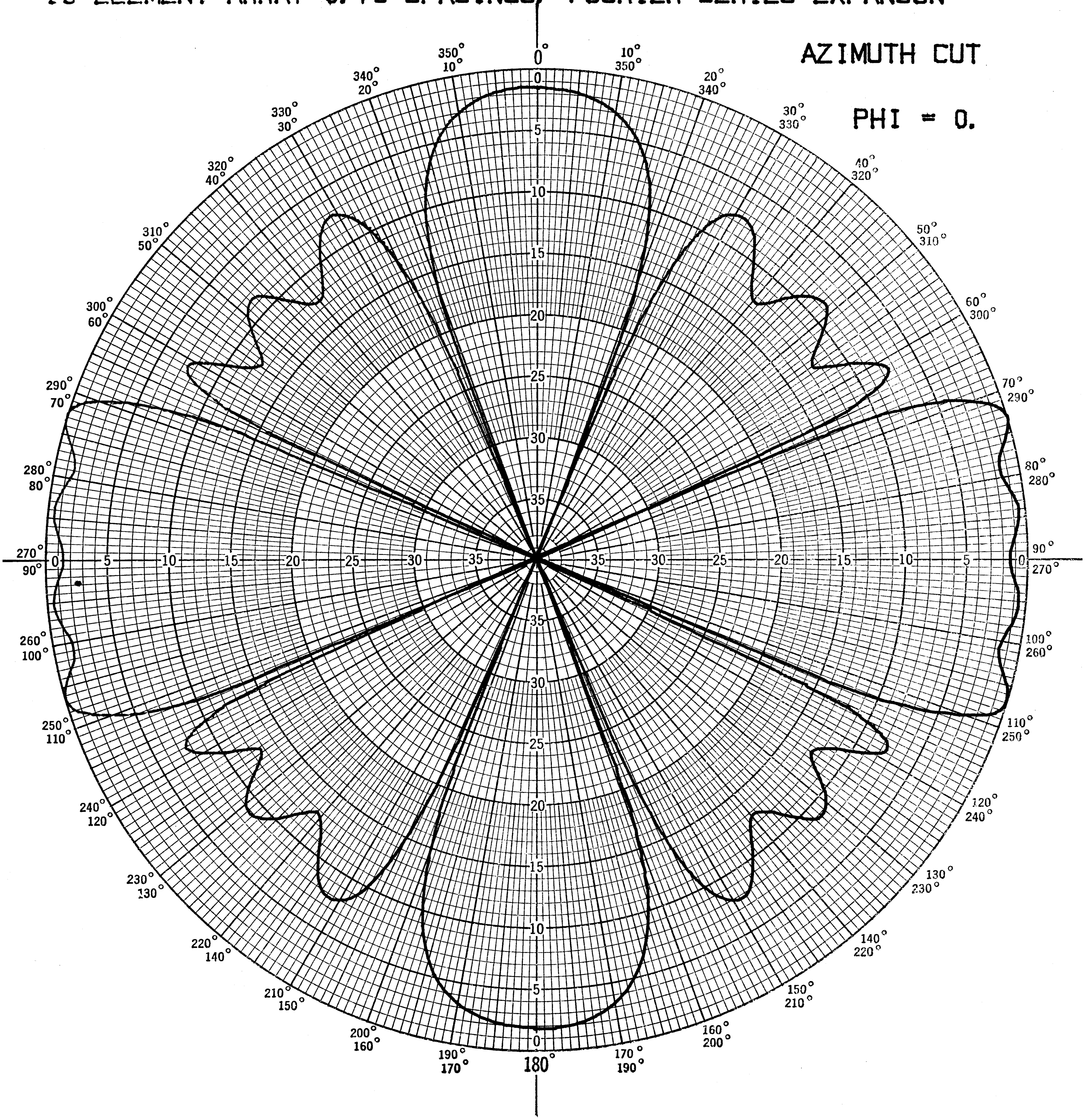
13 ELEMENT ARRAY 0.75 SPACINGS, FOURIER SERIES EXPANSON



harmonics of the distribution are included in the array and the better it matches the aperture distribution.

Suppose we want to scan the beam to 60° using the same $\lambda/2$ spacings, 21 elements, and 45° beamwidth. The beam edges are 37.5° and 82.5°. We can find the coefficients by directly integrating this requirement, but we can use the phase shift between elements, δ , to simplify the problem.

$$180° \,\mathrm{Cos}(37.5°) + \delta \qquad\qquad 180° \,\mathrm{Cos}(82.5°) + \delta$$

$$142.8° + \delta \qquad\qquad 23.49° + \delta$$

We can pick δ to center the beam in ψ space.

$$b = 142.8° + \delta \qquad\qquad -b = 23.49° + \delta$$

$$\delta = -83.15° \qquad\qquad b = 59.65°$$

We can use the formula $\mathrm{Sin}(nb)/(\pi n)$ to find the coefficients of the array.

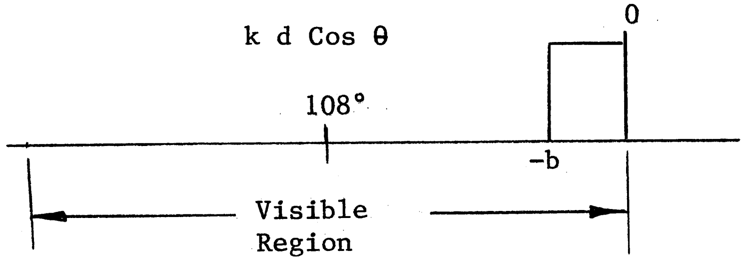
n	a_n	Amplitude	Phase
0	1	0 dB	0°
±1	0.8630	-1.28	∓83.15°
±2	0.4360	-7.21	∓166.30°
±3	0.0060	-44.39	±110.55°
±4	-0.2134	-13.42	∓152.60°
±5	-0.1761	-15.08	±124.25°
±6	-0.0060	-44.39	±41.10°
±7	0.1206	-18.47	±137.95°
±8	0.1111	-19.08	±54.80°
±9	0.0060	-44.39	∓28.35°
±10	-0.0834	-21.57	±68.50°

This array is plotted on page 800 and we can see that the center of the beam is scanned to 60°.

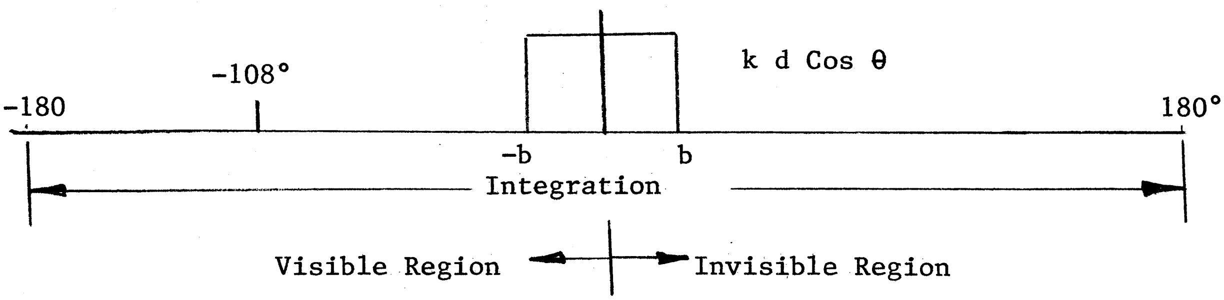
When we scan the beam to endfire, we must approach the problem differently because we must account for the symmetry about $\theta = 0°$. The spacing is limited to less than $\lambda/2$ since grating lobes are present at $\lambda/2$ (see page 85). There is a region of ψ space which is not specified. We can chose the response in this region in any convenient manner. The technique will be demonstrated with an example.

Example: Design a 21 element endfire array with a 90° beamwidth and 0.3λ spacings.

Because the antenna is endfire, we will set $\delta = -kd = -108°$. If we consider the polar diagram, this places the edge of the visible region at the k space origin.



We are free to specify the invisible region. The easiest method is to specify it as the mirror image, so that we have the same problem as above.



Using these diagrams we can solve for b

$$-b = \frac{360°}{\lambda}(0.3\lambda)\,\mathrm{Cos}(45°) - 108°$$

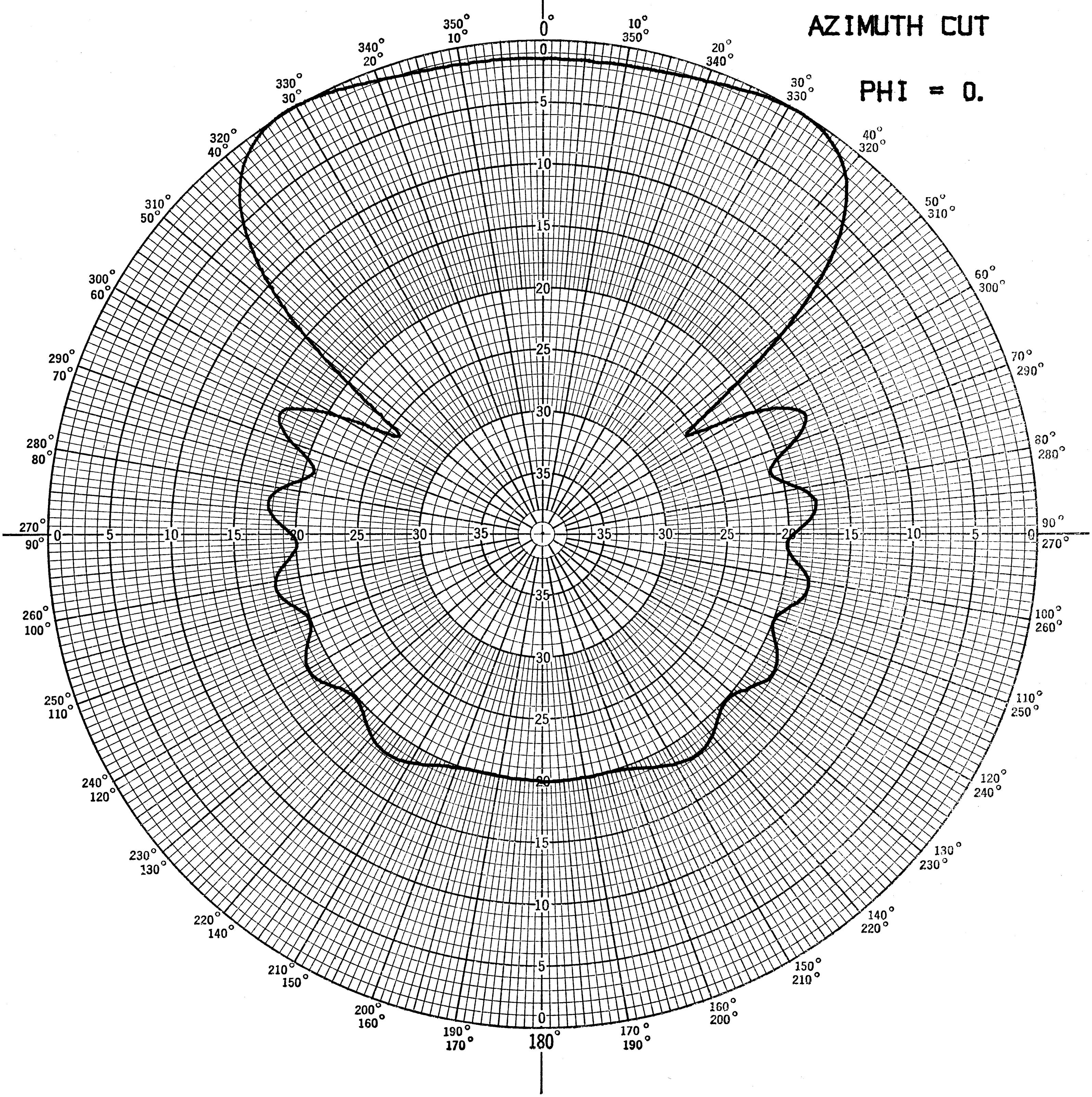
$$b = 31.63°$$

n	a_n	Amplitude	Phase
0	1	0 dB	0°
±1	0.5245	-5.60	108°
±2	0.4465	-7.00	±144°
±3	0.3321	-9.57	±36°
±4	0.2009	-13.94	72°
±5	0.0744	-22.57	180°
±6	-0.0284	-30.95	108°
±7	-0.0945	-20.49	±144°
±8	-0.1196	-18.45	±36°
±9	-0.1075	-19.37	72°
±10	-0.0691	-23.22	180°

The pattern of this array is plotted on page 802. This array does not have the definition that the plot on page 795. Without the beam shaping an array of this size with uniform amplitude has a beamwidth of 44° which is found from the graph on page 353. There is only a limited length of the array which is available for shaping.

The only cases covered in this section have been constant amplitude beams. The method also works equally as well for general shaped beams. If the requirement in k space is curved without sharp corners, then the Fourier series will match more closely the needed aperture distribution. The Fourier series has smaller amplitude harmonics.

21 ELEMENT ARRAY 0.3 SPACINGS, 108 PHASE SHIFT



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WOODWARD-LAWSON SAMPLING

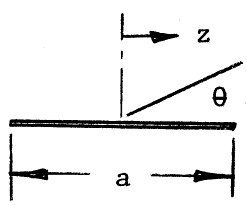
This method is suitable for synthesizing aperture distributions as well as arrays. The array feeding coefficients are found by sampling the aperture distribution. The aperture distribution is found by sampling the desired pattern in k space at even intervals. No integrals are required to find the coefficients.

The technique is based on the scanned pattern of a uniform amplitude distribution. It is convenient to place the nulls of the pattern at integers as was done on page 543 for the Taylor line source.

$$\frac{\sin \pi(U - U_o)}{\pi(U - U_o)}$$

U_o is the maximum of the pattern in U space with the nulls occuring at integer values of $(U - U_o)$ with

$$U = a/\lambda \cos\theta \qquad\qquad U_o = a/\lambda \cos\theta_o$$

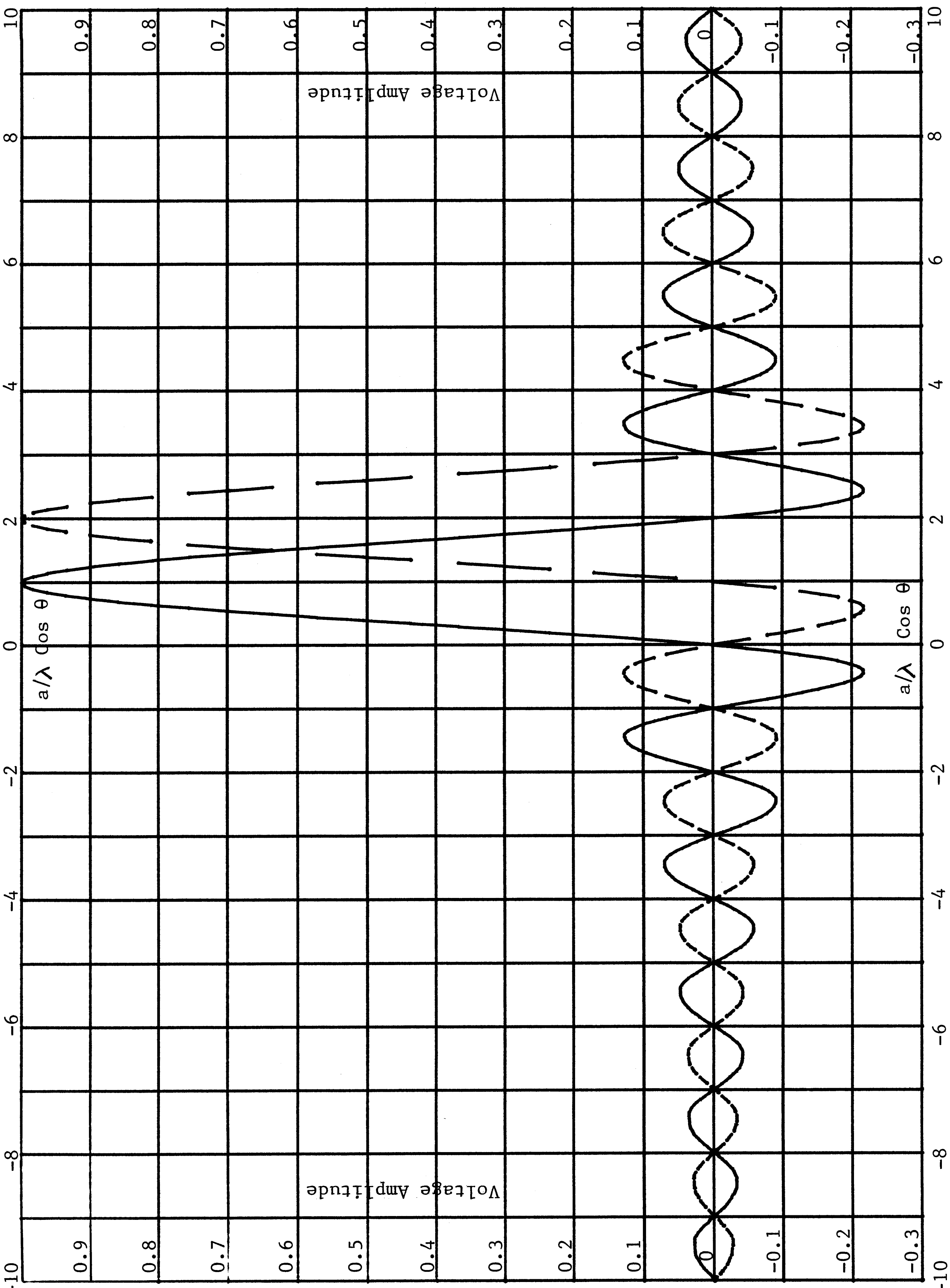


U space is k space divided by π. The boundaries of the visible region are given by ±a in U space. On page 804 are voltage patterns in U space of two different uniform distributions with $U_o = 1$ and $U_o = 2$ for an aperture 10λ long. The nulls of both patterns occur at integer values of U. Notice that the pattern peak of the dashed curve ($U_o = 2$) occurs at one of the nulls of the solid curve ($U_o = 1$). If we only allow integer values of U_o, then the U space pattern at U_o is determined solely by the uniform distribution scanned to U_o. Consider the pattern below U = 0 and above U = 3. The two curves are on opposite sides of zero and tend to cancel when the distributions are added. Each sample point in U space is a uniform distribution which has been scanned. In general, we have $2a/\lambda + 1$ sample points which are independent at integer values of U_o.

To synthesize a desired pattern we must first plot it in U space over the visible region. The pattern is sampled at integer values of U. The aperture distribution is the sum of the uniform distributions scanned to the various angles. When a uniform distribution is scanned to U_o, its aperture distribution is given by

$$E = E_o e^{-j(U_o/a)z}$$

The U space pattern is sampled only at integer values of U_o. The amplitude distribution for an aperture which is an integer number of wavelengths long is given by


Voltage Amplitude
a/λ Cos θ
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
10
8
6
4
2
0
−2
−4
−6
−8
−10

$$E = \sum_{i=-N}^{N} E_i \, e^{-j(i/a)z}$$

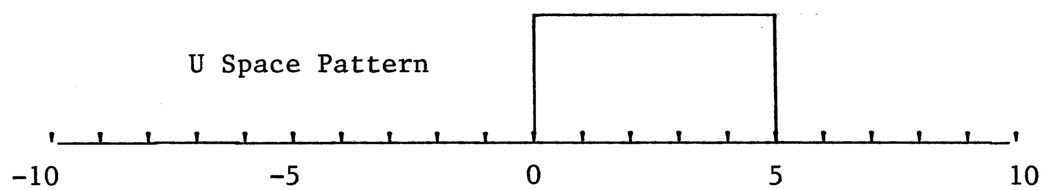
E_i are the samples at integer values of U and $N = a/\lambda$ with N an integer. Array coefficients can only be found easily by sampling the distribution. The U space pattern cannot be sampled at its nulls because they are not usually on the unit circle in the W plane. The Woodward-Lawson method can produce patterns without nulls. See the top of page 766 for an example of a pattern without nulls and its W space zeros distribution.

Example. Design a 20 element array with $\lambda/2$ spacings with a constant beam between $\theta = 60°$ and $\theta = 90°$.

The physical length of the array is 9.5 wavelengths but the effective length is 10λ, see page 789. We first plot this distribution in U space.

$$U_1 = 10 \operatorname{Cos}(60°) = 5 \qquad U_2 = 10 \operatorname{Cos}(90°) = 0$$

Because the length of the aperture is 10λ, the limit of the visible region is U space is ±10.



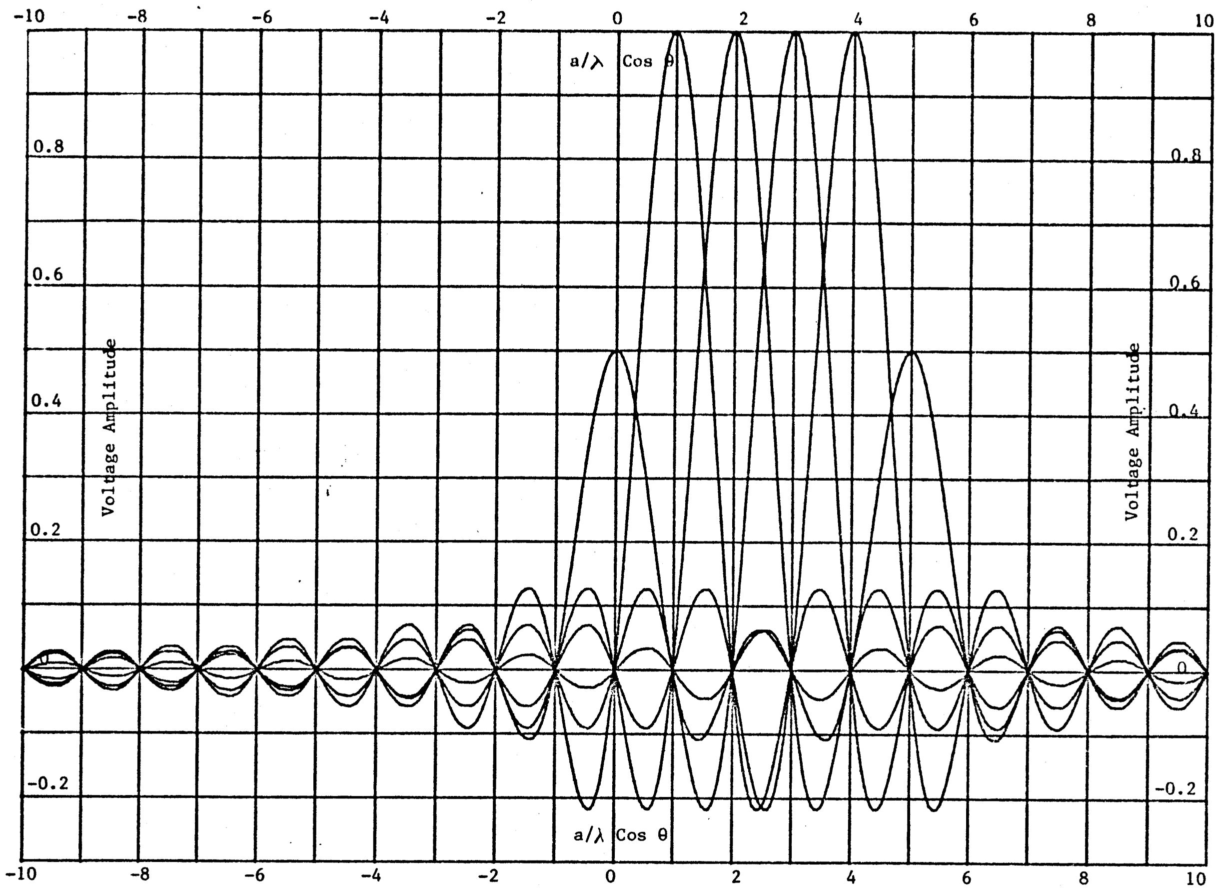
When we sample this distribution, we get the following amplitudes.

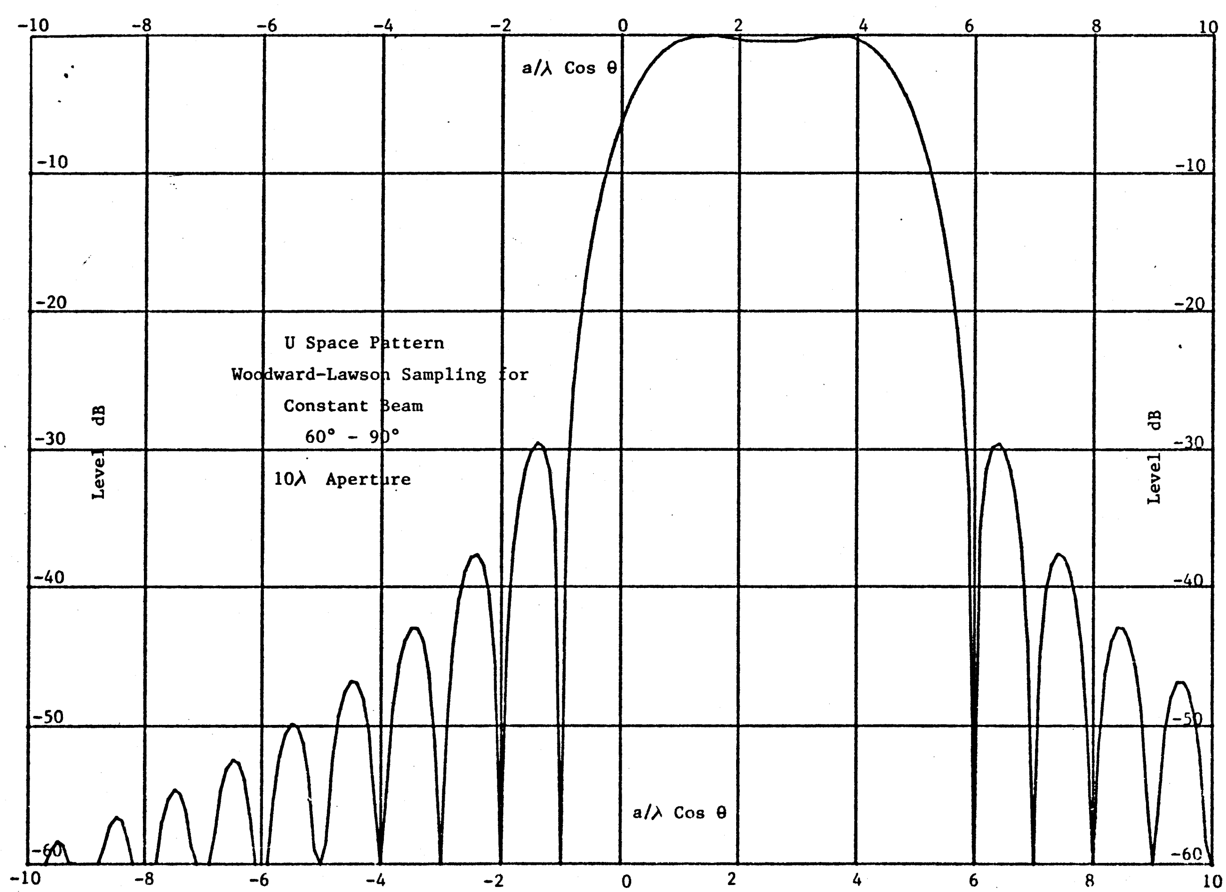
i	E_i	i	E_i	i	E_i
-10	0	-3	0	4	1
-9	0	-2	0	5	0.5
-8	0	-1	0	6	0
-7	0	0	0.5	7	0
-6	0	1	1	8	0
-5	0	2	1	9	0
-4	0	3	1	10	0

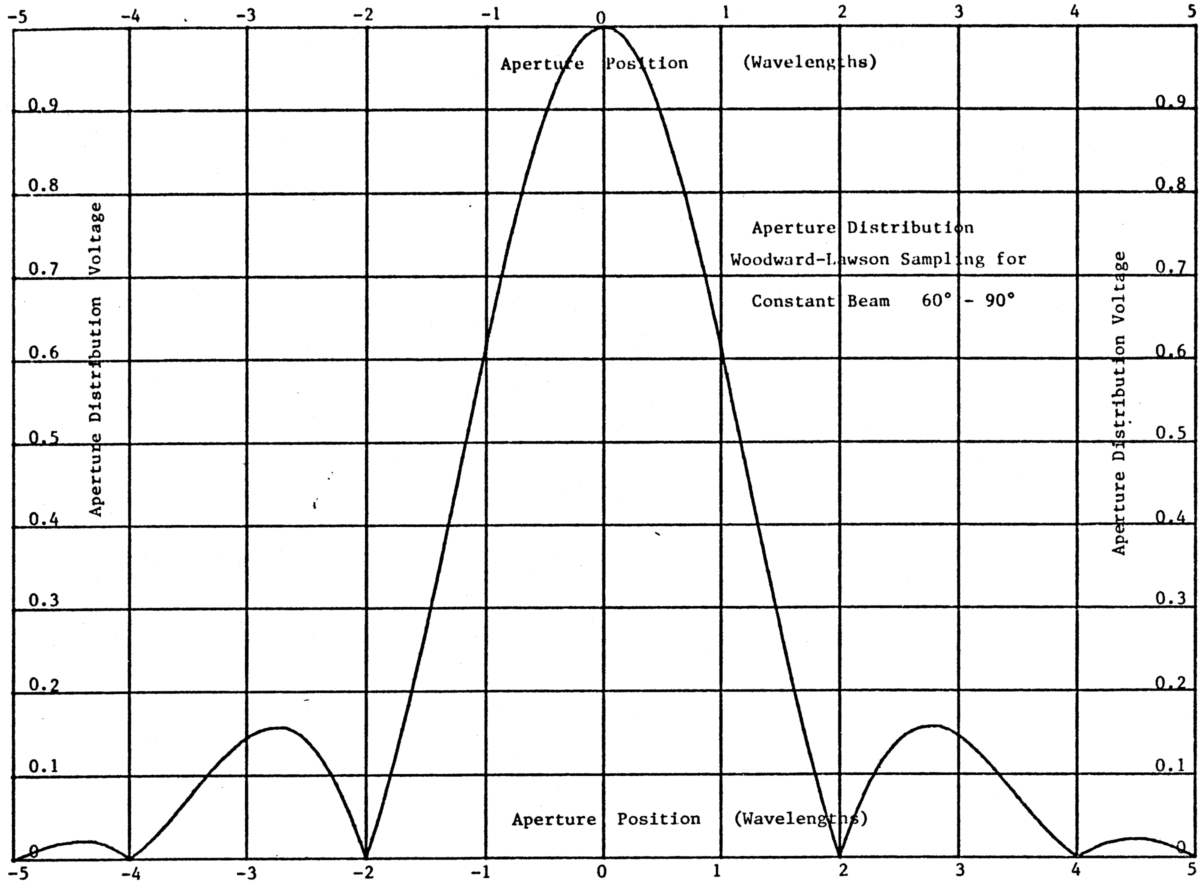
Since the value of the U space pattern at U = 0 and U = 5 can be taken as either 0 or 1 at the discontinuity, we will use the average. A plot showing the various U space patterns of the scanned uniform amplitude distributions are given on the top of page 806. On the bottom is a plot of the summation of these patterns which have been normalized to the peak and converted to dB. The U space pattern is down by 6 dB at the beam edges. The amplitude distribution is found from the formula:

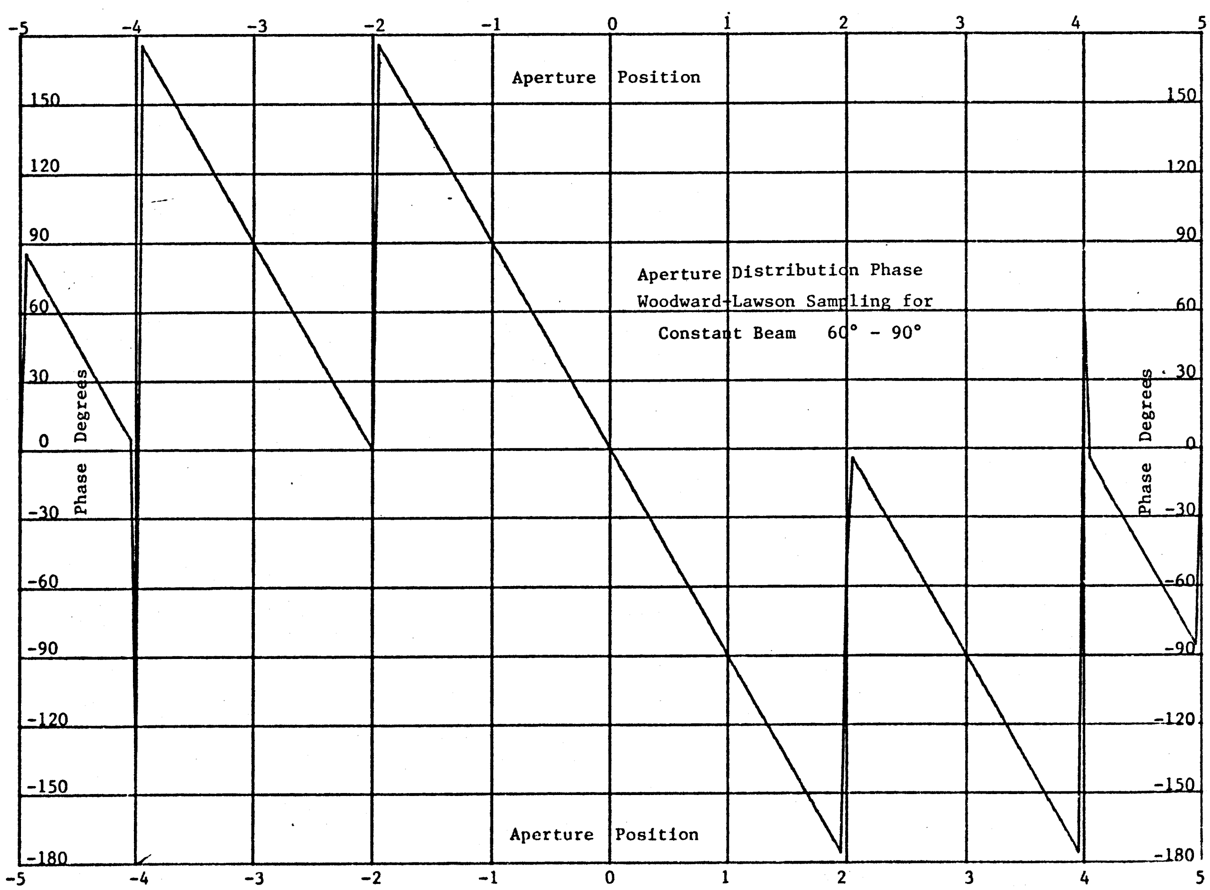
$$0.5 + e^{-jz/a} + e^{-j\,2z/a} + e^{-j\,3z/a} + e^{-j\,4z/a} + 0.5\, e^{-j\,5z/a}$$

The amplitude and phase of this distribution are plotted on page 807. The slope of the phase is negative to scan the beam off broadside.


a/λ Cos θ
Voltage Amplitude
0.8
0.6
0.4
0.2
0
-0.2
-10
-8
-6
-4
-2
0
2
4
6
8
10


a/λ Cos θ
U Space Pattern
Woodward-Lawson Sampling for
Constant Beam
60° - 90°
10λ Aperture
Level dB
-10
-20
-30
-40
-50
-60


Aperture Position (Wavelengths)
Aperture Distribution
Woodward-Lawson Sampling for
Constant Beam 60° - 90°
Aperture Distribution Voltage
Aperture Distribution Voltage
Aperture Position (Wavelengths)


Aperture Position
Aperture Distribution Phase
Woodward-Lawson Sampling for
Constant Beam 60° - 90°
Phase Degrees
Phase Degrees
Aperture Position

The array coefficients can be found by sampling the distribution. After normalizing to the total power, we get the following coefficients.

n	a_n	Amplitude	Phase
∓10	.0077	-42.31 dB	±67.5°
∓9	.0097	-40.28	±22.5°
∓8	.0167	-35.54	±157.5°
∓7	.0597	-24.48	±112.5°
∓6	.0832	-21.60	±67.5°
∓5	.0472	-26.52	±22.5°
∓4	.0658	-23.63	±157.5°
∓3	.2351	-12.57	±112.5°
∓2	.4056	-7.84	±67.4°
∓1	.5125	-5.81	±22.5°

A polar pattern of this array is plotted on page 809.

Suppose we design an array with the same specifications as the example on page 794 using 20 elements and $\lambda/2$ spacings. The Fourier series design is given on page 796 with the pattern on page 797. A constant beam is wanted from 67.5° to 112.5°.

The effective array length is 10 wavelengths. The pattern limits in U space are given by

$$U_1 = 10\ \text{Cos}(67.5°) = 3.82 \qquad U_2 = 10\ \text{Cos}(112.5°) = -3.82$$

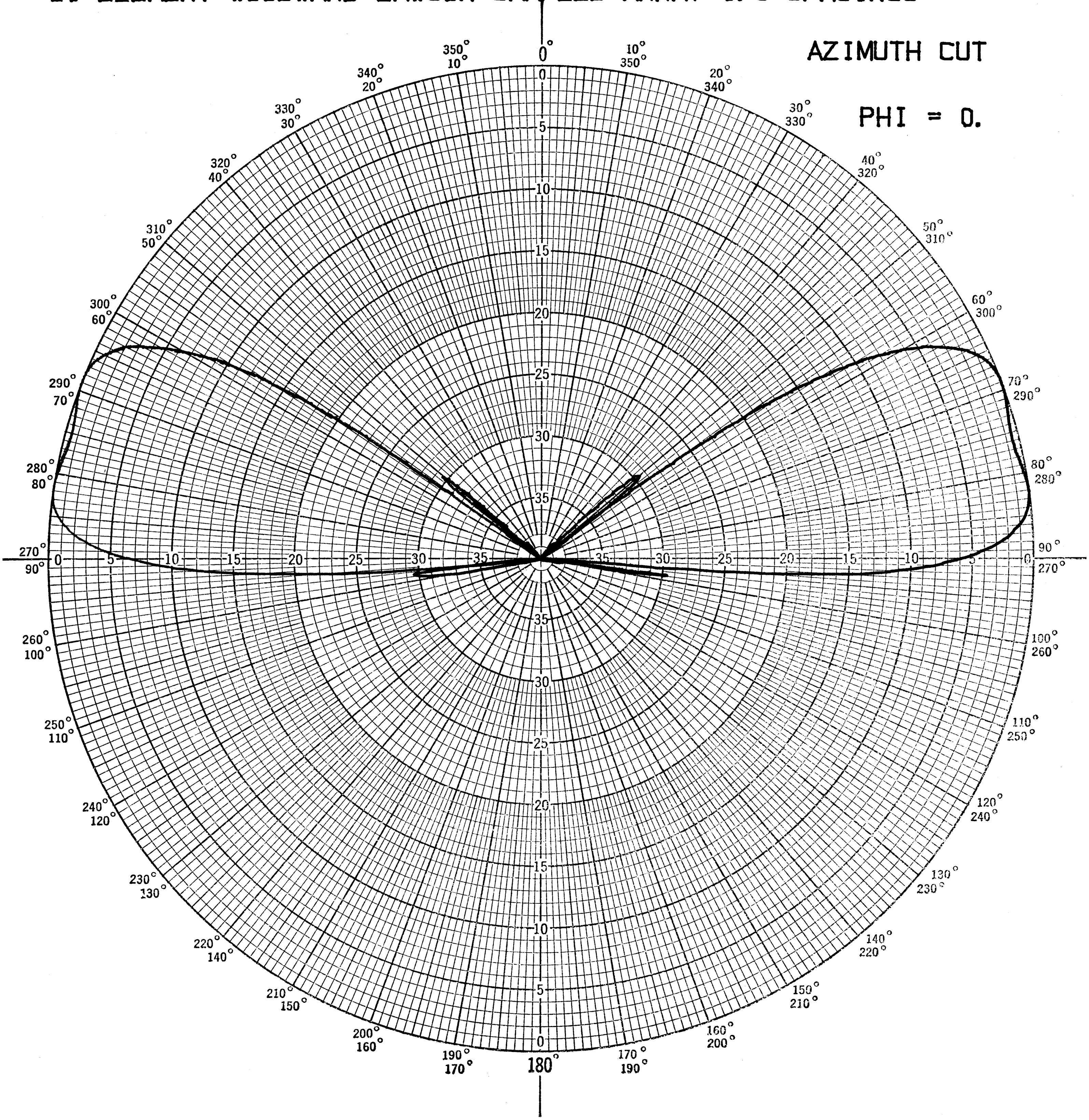
The array sampling at integer values of U gives the following coefficients for the scanned uniform amplitude distributions.

i	E_i	i	E_i	i	E_i
-10	0	-3	1	4	0
-9	0	-2	1	5	0
-8	0	-1	1	6	0
-7	0	0	1	7	0
-6	0	1	1	8	0
-5	0	2	1	9	0
-4	0	3	1	10	0

Because the coefficients are symmetrical about i = 0, the phase of the aperture distribution is either 0° or 180°. The distribution can be written as a sum of cosine terms. When the aperture distribution is sampled, we get the following array coefficients:

n	Amplitude	Phase	n	Amplitude	Phase
±10	-22.82 dB	180	±5	-17.96	180
±9	-43.32	0	±4	-19.57	180
±8	-21.46	0	±3	-21.46	0
±7	-22.46	0	±2	-8.85	0
±6	-31.72	180	±1	-4.99	0

20 ELEMENT WOODWARD-LAWSON SAMPLED ARRAY 0. 5 SPACINGS



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The pattern is plotted on page 811. When comparied to the pattern on page 797, we can see that there are fewer ripples in the pattern response for the Woodward-Lawson design and they are slightly smaller. One problem is that the beam is too narrow. This happens because the sampling was at discrete points in U space. The pattern edges are at ±3.82 but the sampled value at ±4 is zero. The method can only give the proper beamwidth for a constant beam antenna when the beam edges occur at integer values in U space. The point $U_o = 3$ corresponds to

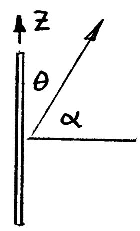
$$\theta = \mathrm{Cos}^{-1}(3/10) = 72.5°$$

Notice that the edge of the ripple occurs ar this angle. The method has difficulty accurately producing an arbitrary beamwidth. It is better suited to a more continuous U space pattern.

A good example of a continuous pattern distribution which can be designed using the Woodward-Lawson sampling technique is a cosecant squared power pattern. When used on the ground with the maximum toward the horizon, the pattern falls off so that an aircraft flying at a constant altitude receives a constant signal.

Design a 20 element array with 0.5 wavelength spacings and a cosecant squared pattern from θ = 20° to θ = 85° with the maximum at 85°.

First convert the cosecant voltage pattern to a U space pattern. The pattern coordinates are given below.



The pattern is given by: $E = \mathrm{Sin}(\alpha_m)/\mathrm{Sin}(\alpha)$ where α_m is the angle of the maximum of the pattern. We can convert this to the normal angle θ.

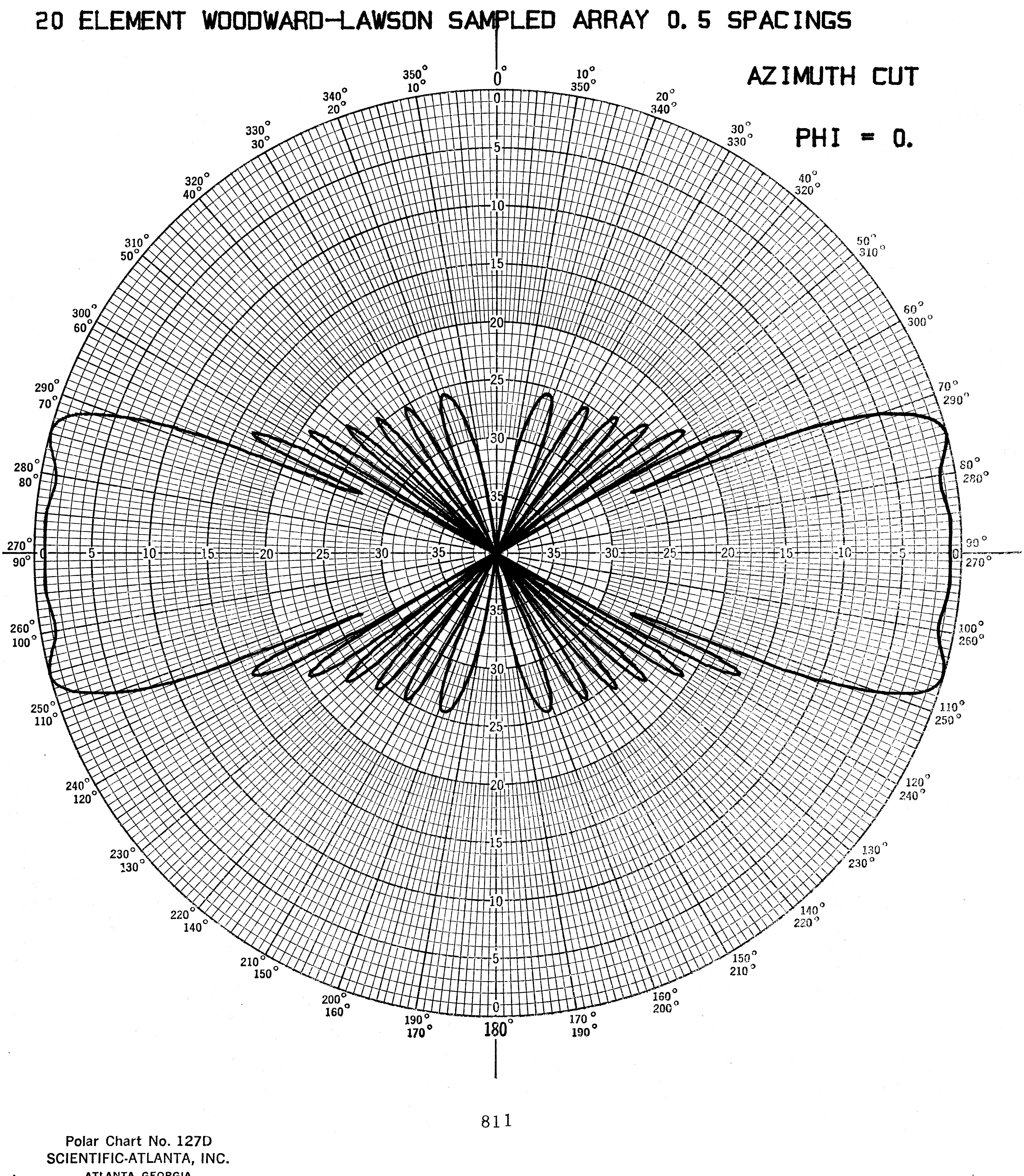
$$E = \mathrm{Cos}(\theta_m)/\mathrm{Cos}(\theta)$$

$$U = a/\lambda \ \mathrm{Cos}\ \theta$$

$$E(U) = U_m/U \qquad U_m = 10\ \mathrm{Cos}(85°) = 0.8716$$

The U space pattern is the 1/X function which must be sampled at integer values of U.

i	E_i	i	E_i	i	E_i
0	0	4	0.2179	8	0.1089
1	0.8716	5	0.1743	9	0.0968
2	0.4358	6	0.1453	10	0
3	0.2905	7	0.1245		


20 ELEMENT WOODWARD-LAWSON SAMPLED ARRAY 0.5 SPACINGS
AZIMUTH CUT
PHI = 0.
Polar Chart No. 127D
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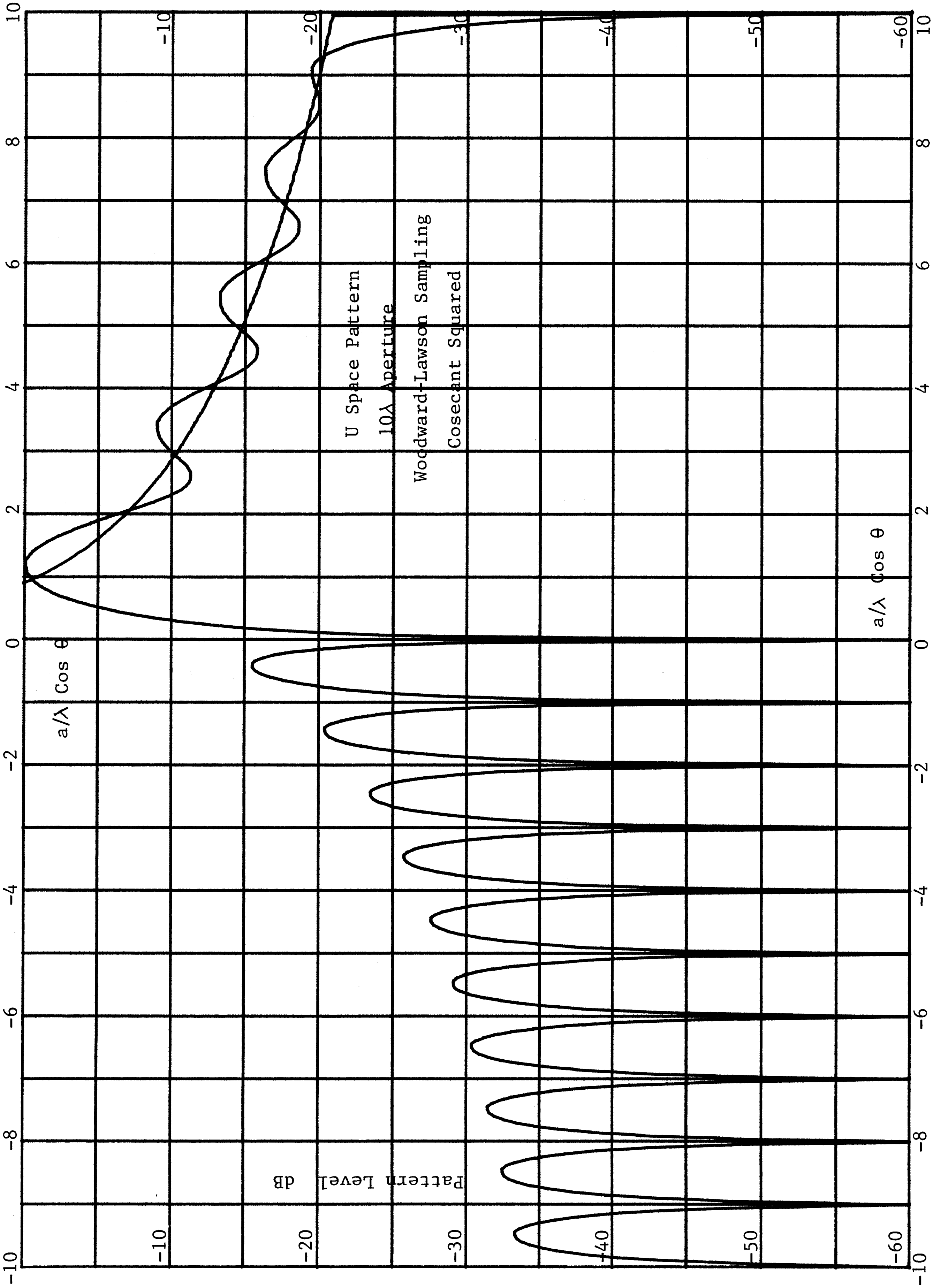
The U space pattern of this sampling is drawn on page 813 along with the desired pattern. The ripple in the pattern is typical of a cosecant synthesized pattern and are about the same size for an aperture twice as long. The amplitude and phase of the aperture distribution are plotted on page 814. The phase has a negative slope to scan the beam above broadside.

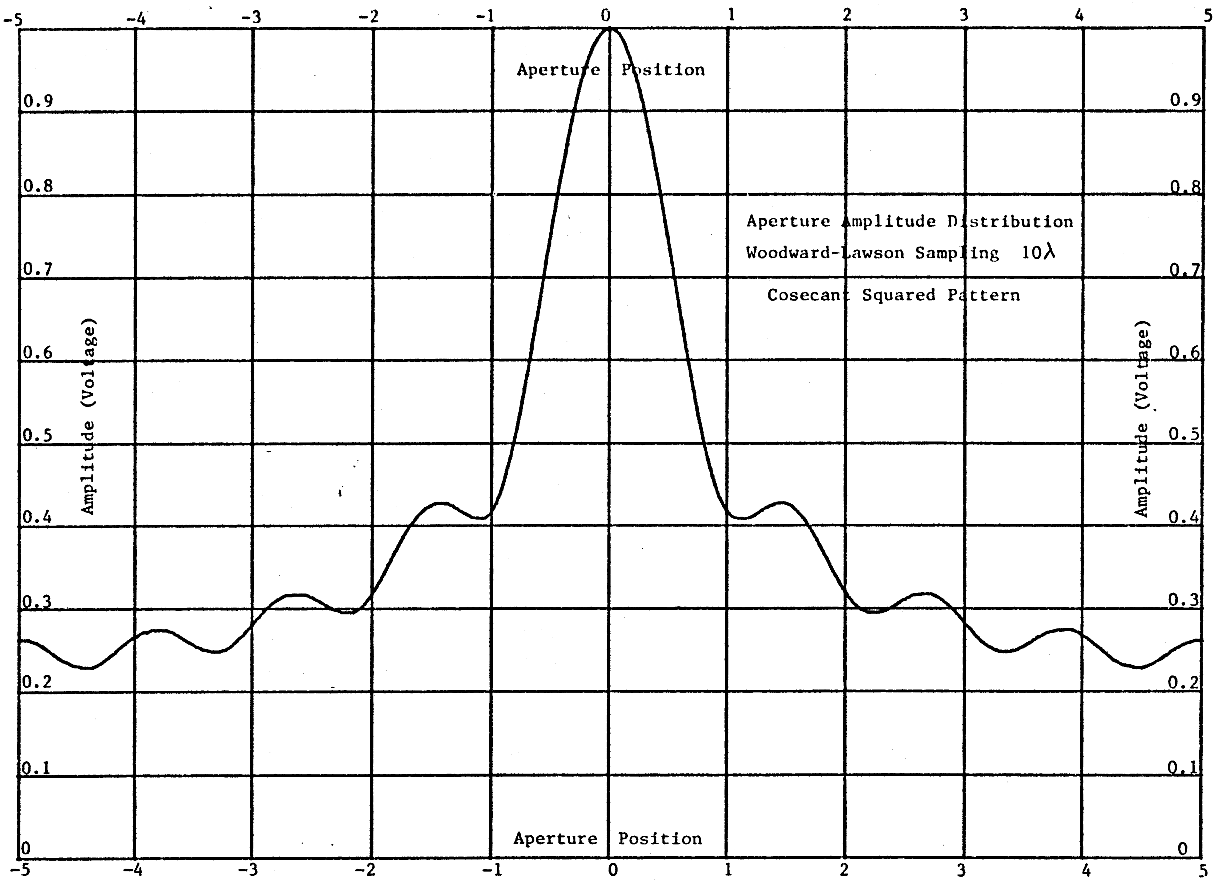
The array coefficients are found by sampling the aperture distribution.

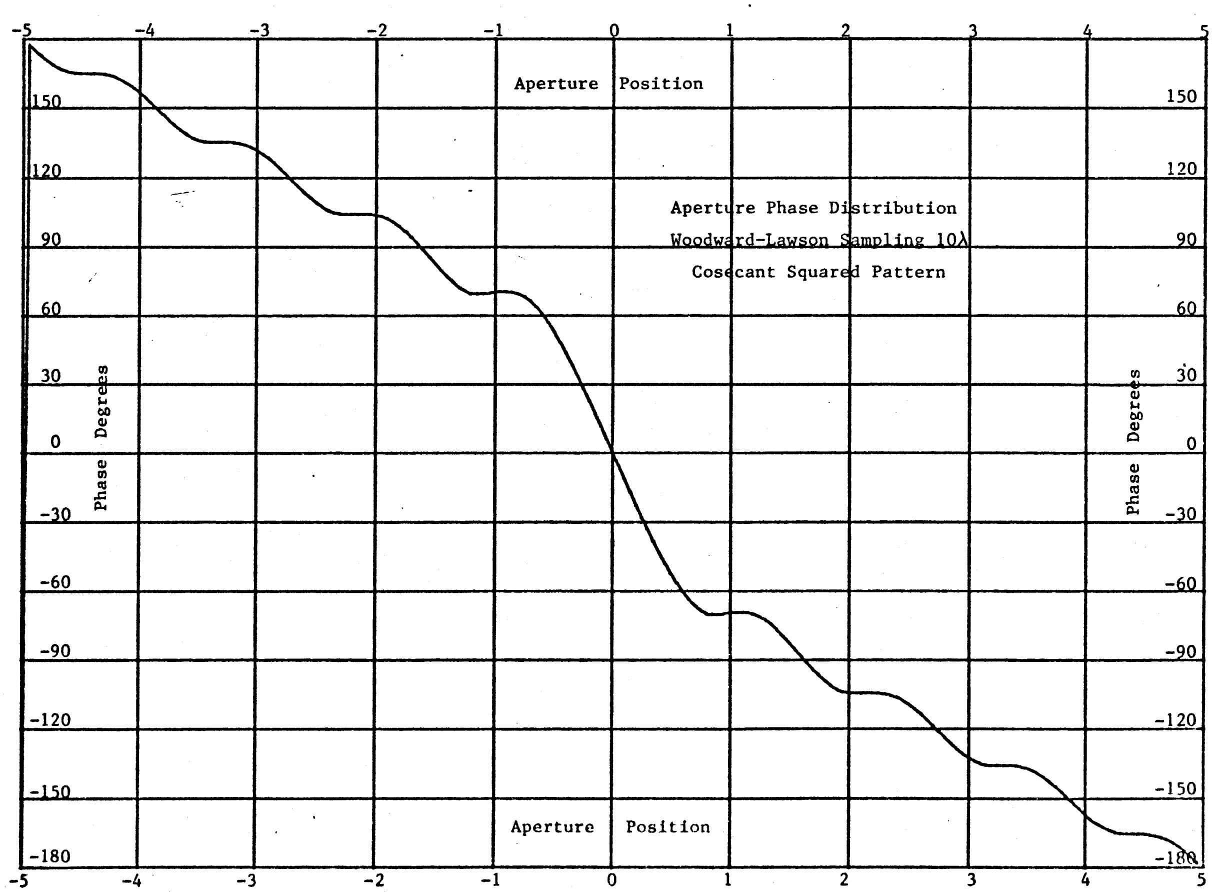
n	Amplitude	Phase
∓1	-6.47 dB	±28.12°
∓2	-11.18	±68.28°
∓3	-13.47	±70.93°
∓4	-14.16	±96.56°
∓5	-16.47	±104.14°
∓6	-15.90	±121.24°
∓7	-17.90	±135.40°
∓8	-17.09	±144.92°
∓9	-18.26	±164.35°
∓10	-17.91	±169.31°

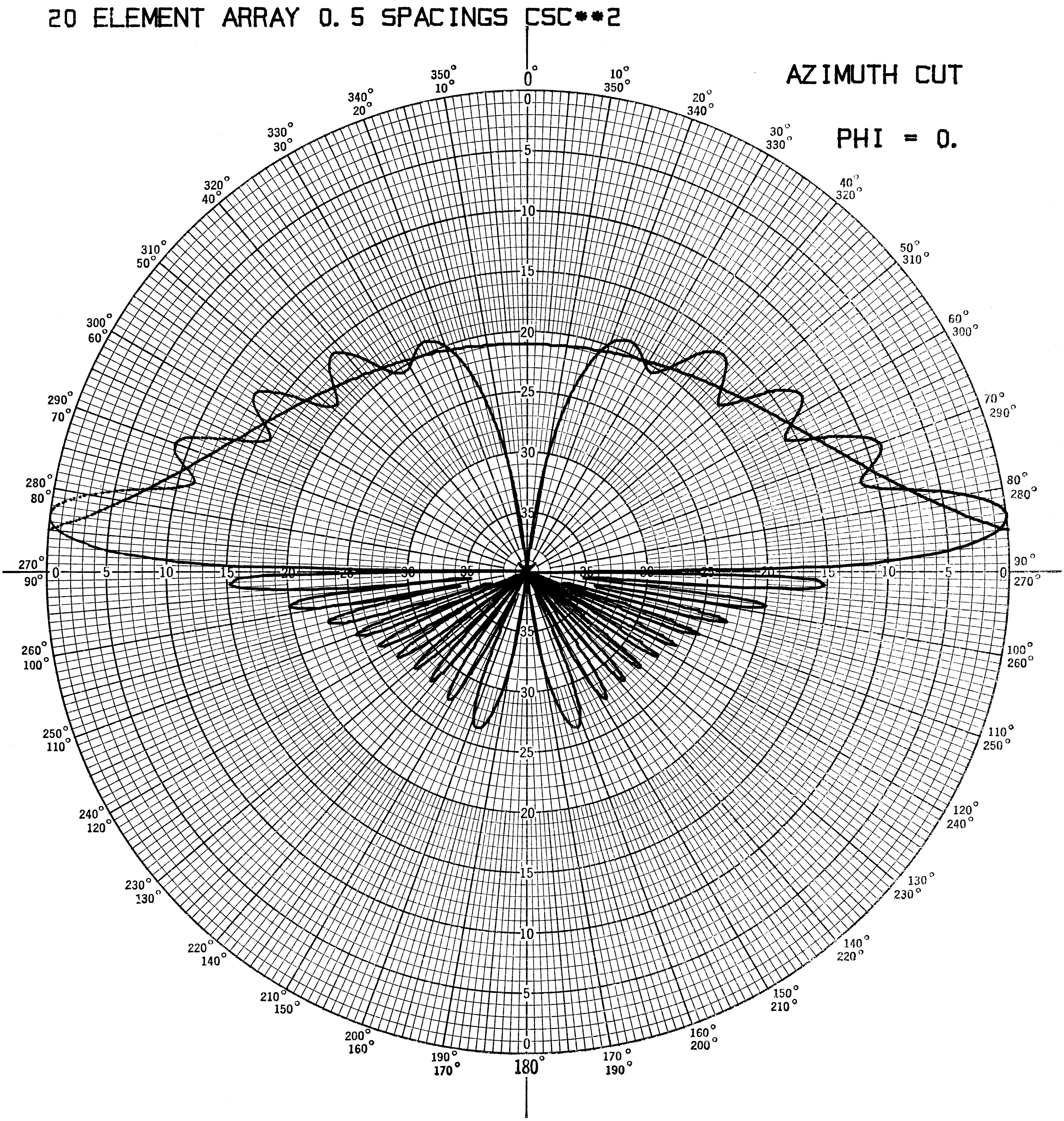
The polar pattern of the array is drawn on page 815 along with a cosecant squared pattern which peaks at $\theta = 85°$.

The Woodward=Lawson technique is able to design an aperture distribution without integrating the k space pattern which is required by the Fourier series method. Only a simple sampling is required to achieve a closed form solution. The aperture distribution can be sampled to find the array coefficients.


U Space Pattern
10λ Aperture
Woodward-Lawson Sampling
Cosecant Squared
Pattern Level dB
a/λ Cos θ
a/λ Cos θ
−10 −8 −6 −4 −2 0 2 4 6 8 10
−10 −20 −30 −40 −50 −60


Aperture Position
Aperture Amplitude Distribution
Woodward-Lawson Sampling 10λ
Cosecant Squared Pattern
Amplitude (Voltage)
-5 -4 -3 -2 -1 0 1 2 3 4 5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9


Aperture Position
Aperture Phase Distribution
Woodward-Lawson Sampling 10λ
Cosecant Squared Pattern
Phase Degrees
-5 -4 -3 -2 -1 0 1 2 3 4 5
150 120 90 60 30 0 -30 -60 -90 -120 -150 -180



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