The purpose of this course is to work for an understanding of antenna theory and not to give an exacting mathematical treatment. We must start with electromagnetics. If we start with Maxwell's equations and begin deriving, then most of the feel for what is happening is lost in mathematics. The starting place of the material will be too elementary for some of you, but it is good to review basic principles once in awhile. If it is all new, then it will move along too fast.

## ELECTRIC FIELD

Most of the time we will be talking about the electric field. Unlike voltage, it is valid regardless of the frequency. The electric field is defined by the force on a charged particle. If there is a force F on a charged particle, then the electric field is the force divided by the charge.

$$E = F/Q \qquad \text{(units: volts/meter or newtons/coulomb)}$$

The electric field is a vector quantity like force (in the same direction as the force, if Q is positive).

## VOLTAGE

The voltage can be defined for electrostatics, but we can use it in a restricted sense for time varying fields. Since it is not always strictly correct, we have to be careful when using it. Assume that the field is constant over a small length, then the incremental voltage between the ends of the length is given by:

$$V = -E\,L$$

Where **E** is in the same direction as the length. To get the voltage between two points, we sum these uniform electric fields times the lengths.

Note the material given below which is bracketed is not strictly necessary, but is included for completeness. In general this is true.

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In electrostatics the voltage is defined as a line integral.

$$V = -\int \bar{E} \cdot d\bar{\ell}$$

This is valid if the integral is the same no matter what path is taken between the two points. If this is true, then the electric field can be found from the gradient of the voltage.

$$\bar{E} = -\text{grad } V = -\nabla V$$

$$\bar{E} = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right] \qquad \text{in rectangular coordinates}$$

In general voltage remains a valid term for TEM transmission lines such as coax lines.

The voltage is valid when:

$$\oint \bar{E} \cdot d\bar{\ell} = 0$$

but in general.

$$\oint \bar{E} \cdot d\bar{\ell} = -\iint_s \frac{d\bar{B}}{dt} \cdot d\bar{s}$$

This is Faraday's law which gives the induced voltage in a loop from the rate of change of the integral over the surface of the magnetic flux density, B.

From vector analysis a vector field derived as the gradient of a scalar function has zero curl, but in general the curl of the electric field is not zero.

$$\text{curl } \bar{E} = \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

In rectangular coordinates the curl of a vector A is given by:

$$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \bar{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \bar{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + a_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

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ELECTRIC FLUX DENSITY

The second quantity is the electric flux density. This is related to the electric field:

$$\bar{D} = \epsilon \bar{E} \qquad \text{(coulombs/(meter)}^2\text{)}$$

where $\epsilon$ is the permittivity. In an isotropic medium (the properties are the same in all directions) D is in the same direction as E and $\epsilon$ is a scalar quantity.

$$\epsilon_o = 8.854 \times 10^{-12} \text{ farads/meter}$$

is the permittivity of free space. We also use the relative permittivity so that:

$$\epsilon = \epsilon_r \epsilon_o$$

$\epsilon_r$ is also called the dielectric constant.

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The integral of the electric flux density over a closed surface is related to the charge enclosed by the surface. Gauss's law: The surface integral of the electric flux density over a closed surface is equal to the charge enclosed by the surface.

 

$$\oint_S \bar{D}\cdot d\bar{S} = Q = \iiint_V \rho \, dv$$

Where $\rho$ is the charge density in the region and the second integral is over the volume enclosed by the surface. Since from Gauss's Theorem,

$$\oint_S \bar{A}\cdot d\bar{S} = \iiint_V \nabla\cdot\bar{A}\, dv = \iiint_V \operatorname{div}\bar{A}\, dv$$

The div $\bar{A}$ is the divergence of the vector $\bar{A}$. In rectangular coordinates the diverence is given by:

$$\operatorname{div}\bar{A} = \nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The differential form of Gauss's law can be found.

$$\oint_S \bar{D}\cdot d\bar{S} = \iiint_V \nabla\cdot\bar{D}\, dv = \iiint_V \rho\, dv$$

Since the last two integrals are valid over any volume, then the integrands are equal and the differential form of Gauss's law is found.

$$\nabla\cdot\bar{D} = \rho$$

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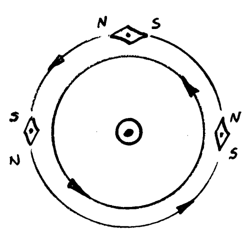
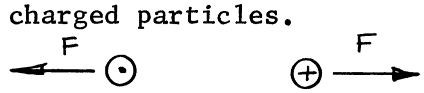
MAGNETIC FIELD

It was discovered that a wire carrying current would deflect a compass needle. If a wire is carrying current out of the page in the figure below, then there is a magnetic field surrounding it. The compass needle always orientates itself in the direction shown. Note that there are closed

Current is out of the paper

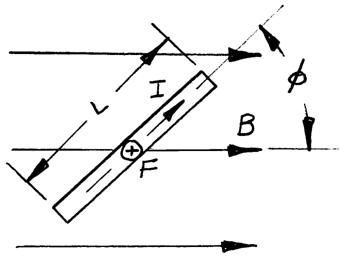



circular loops around the wire. The direction of the magnetic field is indicated by the direction of "north" of the compass needle. It was also discovered that there are forces between wires carrying current similiar to the forces between charged particles.




MAGNETIC FLUX DENSITY

The flux density is related to the force on a wire with a current through it. Take a straight current element in a uniform magnetic flux density. The force on the wire is proportional to the length of the wire, the magnetic flux density, the current in the wire, and the sine of the angle between the wire and the magnetic flux density. The force is in a direction which is orthogonal (perpendicular) to both the wire and the magnetic flux density.




F is into the paper.

B is the Magnetic Flux Density

$$B = \frac{F}{IL \sin\phi}$$

---

The magnetic flux density is always in closed curves. This property of the vector field is called solenoidal and is expressed by the integral:

$$\oint_S \bar{B} \cdot d\bar{S} = 0$$

From Gauss's Theorem this can be expressed as:

$$\oint_S \bar{B} \cdot d\bar{S} = \iiint_V \nabla \cdot \bar{B}\, dv = 0$$

Since this holds for any volume, then the integrands are equal.

$$\nabla \cdot \bar{B} = 0$$

The force on a differential length wire carrying current can be expressed in vector notation.

$$d\bar{F} = I(d\bar{\ell} \times \bar{B})$$

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MAGNETIC FIELD

The term magnetic field has been loosely used. It is defined in terms of the magnetic flux density.

$$\bar{H} = \bar{B}/\mu \qquad \text{(ampere/meter)}$$

Where $\mu$ is called the permeability of the medium. The permeability of free space is defined as:

$$\mu_0 = 4\pi \times 10^{-7} \qquad \text{(henry/meter)}$$

The relative permeability can be defined like the relative permittivity (dielectric constant).

$$\mu = \mu_r \mu_0$$



 

The line integral of the magnetic field around a closed curve is equal to the current flowing through the surface defined by the curve (Ampere's law).

$$\oint \bar{H} \cdot d\bar{\ell} = I = \iint_S \bar{J} \cdot d\bar{S}$$

Where $\bar{J}$ is the current density. Since this integral is not zero, the magnetic field vector cannot be defined as a gradient of a scalar function in space. The LINE integral can be related to a SURFACE integral using Stoke's Theorem.

$$\oint \bar{H} \cdot d\bar{\ell} = \iint_S \nabla \times \bar{H} \cdot d\bar{S}$$

Since the two SURFACE integrals can be equated over any SURFACE, the integrands are equal and a differential form of Ampere's law can be found.

$$\nabla \times \bar{H} = \bar{J}$$

We will have to include the time derivative of the electric flux density to get the final form.

TIME VARYING FIELDS

When the electric or magnetic fields vary with time, then the two kinds of fields become interrelated and the one does not exist without the other. It was discovered that a current is induced in a closed circuit if the magnetic flux linking the circuit is changing with time. The magnetic flux is the average magnetic flux density times the area of the circuit. The magnetic flux has the dimensions: webers. The magnetic flux density, B, has the dimensions: weber/(meter)$^2$. The current induced in a loop produces a magnetic flux density. The current induced is in a direction to produce a magnetic flux density which reduces the rate of change of the magnetic flux (Lenz law). Since any loop has some resistance, there is a voltage induced around the loop. Faraday's law states: the total emf (voltage) induced in a closed circuit is equal to the time rate of decrease of the total magnetic flux linking the circuit.

The emf induced by the changing magnetic flux is the line integral of the electric field around the circuit.

$$EMF = \oint \bar{E} \cdot d\bar{\ell}$$

The magnetic flux density is the surface integral of the magnetic flux density.

$$\psi_m = \iint_S \bar{B} \cdot d\bar{S}$$

The induced emf is the time derivative of the magnetic flux.

$$V = -\frac{d}{dt}\iint_S \bar{B}\cdot d\bar{S}$$

In this case the magnetic flux density could be changing or the surface over which the integral is taken, such as a coil rotating in a constant magnetic flux density. If a fixed surface is picked, then the following equation is obtained:

$$\oint \bar{E}\cdot d\bar{\ell} = -\iint_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

Using Stoke's Theorem the line integral can be converted to a surface integral.

$$\oint \bar{E}\cdot d\bar{\ell} = \iint_S (\nabla \times \bar{E})\cdot d\bar{S}$$

The differential form of Faraday's law can be found by considering an arbitrary surface.

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Note that this is an arbitrary loop and not necessarily a loop of wire.

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If we pick an arbitrary loop, then the induced emf (voltage) must be related to the electric field instead of a voltage because there are no wires in an arbitrary loop. This will be the sum of the electric field in the direction of the lengths times the incremental lengths (a line integral). Using Faraday's law over an arbitrary loop the time changing magnetic fields (magnetic flux density) are equated to an electric field.

INDUCTANCE

The inductance is defined as the magnetic flux stored in a circuit divided by the current flowing through it. We can find the circuit relationship of an inductor using Faraday's law. The voltage across an inductor is equal to the time rate of change of the magnetic flux stored in it.

$$L = \psi_m / I$$

Where $\psi_m$ is the magnetic flux, L is the inductance, and I is the current flowing through it. The voltage across it is given by Faraday's law.

$$V = \frac{d\psi_m}{dt}$$

From the equation above the magnetic flux stored is given by:

$$\psi_m = LI$$

Hence the voltage across an inductor is given by:

$$V = L\frac{dI}{dt}$$

## DISPLACEMENT CURRENT

The idea of the displacement current is a contribution of Maxwell and ties the time changing electric field to the magnetic field. The easiest way to explain displacement current is by considering a capacitor. A capacitor stores equal and opposite charges Q on the two plates for a given voltage, V, across it.

$$Q = C V$$

Where C is the capacitance in farads. We define current as the time rate of change of charge (ampere = coulomb/ second). From the conservation of charge, the current through a capacitor is the time rate of change of the charge stored in it.

$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

Which is the circuit equation of the capacitor. Assume a parallel plate capacitor and neglect the fringing fields or include them in an effective area, then the capacitance is given by:

$$C = \frac{\epsilon A}{d}$$

Where A is the effective area of the plates, d is the distance between them, and $\epsilon$ is the permittivity of the medium between the plates. In this case the voltage between the plate is the electric field in the center of the plates where it can be assumed to be constant, times the separation, d. Substituting the above relations into the equation above, the equation for the current through a capacitor becomes:

$$V = Ed \qquad I = \frac{\epsilon A d}{d}\frac{dE}{dt} = \epsilon A\frac{dE}{dt}$$

But the quantity $\epsilon$ E equals the electric flux density.

$$I = A\frac{dD}{dt}$$

The current is equal to the time rate of change of the electric flux, since the electric flux is equal to the electric flux density times the area. Since this a current, it will induce a magnetic field. The important thing is that this is a current between the plate of the capacitor where there is no flow of charges.

---

The electric flux is defined by the surface integral:

$$\psi_e = \iint_S \bar{D}\cdot d\bar{S}$$

From considering the capacitor above, there is a displacement current.

$$I_d = \iint_S \frac{\partial \bar{D}}{\partial t}\cdot d\bar{S}$$

This current must be included in Ampere's law.

$$\oint \bar{H} \cdot d\bar{\ell} = \iint_S \bar{J} \cdot d\bar{s} + \iint_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$$

The line integral can be converted to a surface integral by using Stoke's Theorem.

$$\oint \bar{H} \cdot d\bar{\ell} = \iint_S \nabla \times \bar{H} \cdot d\bar{s}$$

The integral can be defined over any surface and the differential equation of Maxwell's Ampere's law is found.

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

This is the final Maxwell's equation.

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Using Faraday's law and Maxwell's extension of Ampere's law, it is seen that for time varying magnetic or electric fields, that both fields will be present. Each one induces the other.

Two circuit devices have been introduced that store energy. The inductor is a device or a property of a circuit which stores energy in a magnetic field. The capacitor stores energy in an electric field. Circuits will store energy in both electric and magnetic fields. When we measure the input impedance of a device such as an antenna, it may measure capacitive or inductive. This is a measure of which field is storing the most energy. As the frequency changes the energy will cycle between the inductive energy (magnetic field storage) and capacitive energy (electric field storage) as the predominate means of storing energy.

 

## RADIATION

Before we can talk about antennas, we have to consider radiation. We all know that an antenna radiates, but what is it? Maxwell showed that time varying electric and magnetic fields travel through space unconnected to wires, currents, or charges. To be sure, they must be the initial source; but after the waves are generated, they fly off into space. Remember that for time varying fields, the electric and magnetic fields are interconnected. It is these properties which lead to electromagnetic waves. D.C. fields are always connected to the sources that generate them; they do not propagate.

If we take the two Maxwell's equations involving the curl of the electric field and the curl of the magnetic field, we can combine these equations. The result of the combination is a partial differential equation of only the electric or magnetic fields with side relations connecting the electric and magnetic fields. I am not going to do this here. The solution of the equation is a function of time and a space coordinate in the direction of propagation. For example:

$$E = a\, f(t - z/c) + b\, f(t + z/c)$$

Where $c = 1/\sqrt{\mu\epsilon}$ which is the velocity of light. The function f is any function which is twice differentiable; z is the space coordinate in the direction of propagation; and a and b are constants.

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The following is a derivation of the wave equation from Maxwell's equations. Take the two curl equations in free space (no charges or currents).

$$\nabla \times \bar{E} = -\mu \frac{\partial H}{\partial t} \qquad \nabla \times \bar{H} = \epsilon \frac{\partial E}{\partial t}$$

If we take the curl of the first equation we get:

$$\nabla \times \nabla \times \bar{E} = -\mu\, \nabla \times \frac{\partial H}{\partial t}$$

Since the time derivative has no effect on the space coordinates of the curl, then the differentiations can be interchanged.

$$\nabla \times \nabla \times \bar{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \bar{H})$$

Now if we substitute the second Maxwell curl equation for the magnetic field, we can get an equation which is only a function of the electric field.

$$\nabla \times \nabla \times \bar{E} = -\mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$

We can use the following vector identity to expand the curl curl E.

$$\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$$

The divergence of the electric flux density is zero since we assumed a source free space.

$$\nabla \cdot \bar{D} = 0 \qquad \epsilon(\nabla \cdot \bar{E}) = 0$$

The function $\nabla^2 \bar{E}$ in rectangular coordinates is given as:

$$\nabla^2 \bar{E} = \bar{a}_x \nabla^2 E_x + \bar{a}_y \nabla^2 E_y + \bar{a}_z \nabla^2 E_z$$

If we only allow a component in the x direction, then we have a simple second order partial differential equation.

$$\frac{\partial^2 E_x}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

We could do the same thing with the magnetic field which I will leave as an exercise. By substitution we can show that the above equation is a solution of the partial differential equation.

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Maxwell's equations predicted this in 1873 but it was not until 1888 that Hertz was able to show experimentally that electromagnetic waves exist.

Consider the function: $f(t - z/c)$ at time equal zero and the space coordinate $z$ equal zero. It has a value $f_o$. Now consider some time later, $t_o$; the function will have the same value $f_o$ if the following is true.

$$t_o - z/c = 0$$

Which can be solved.

$$z = t_o c$$

This says that the value of the function, $f(t - z/c)$ at time zero and space coordinate zero has moved to a new space coordinate: $z = t_o c$ after a time $t_o$. The value has propagated through space at the velocity of light. The solution of the partial differential equation gives two waves propagating in the plus $z$ and minus $z$ directions.

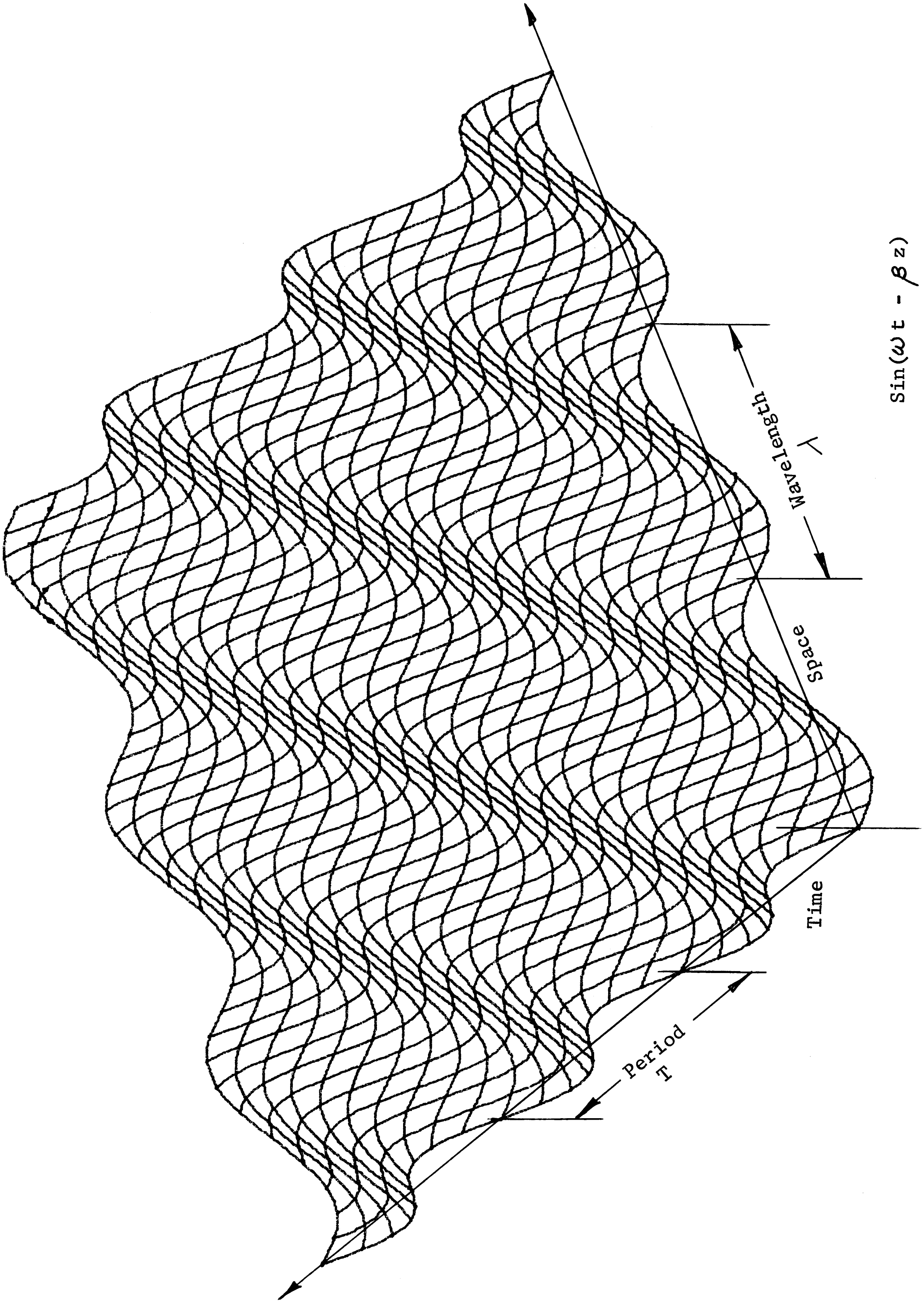
$f(t - z/c)$ propagates in the positive $z$ direction

$f(t + z/c)$ propagates in the negative $z$ direction

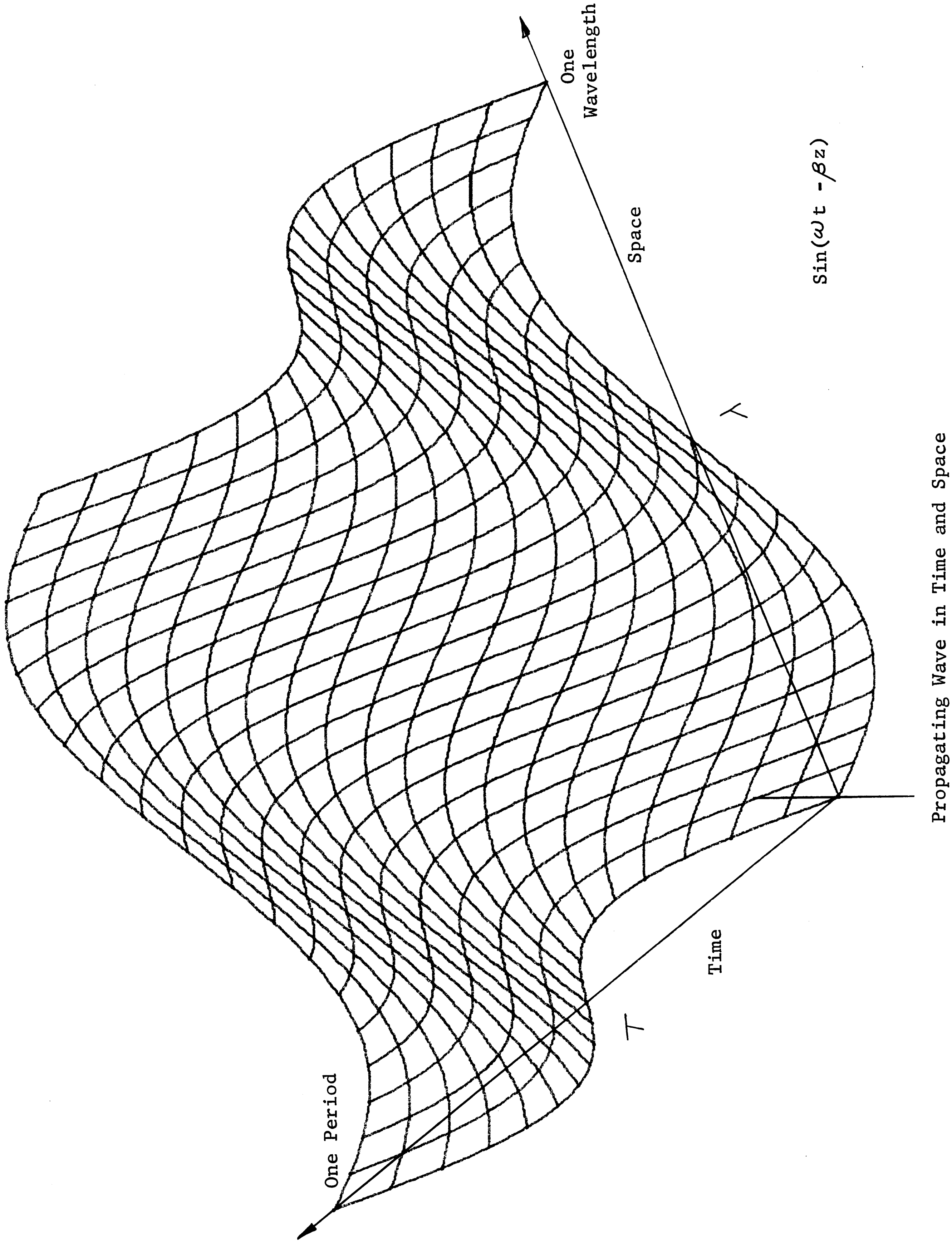
In this course we will only consider sinusoidal functions. The cosine and sine are twice differentiable so they are solutions to the wave equation. The solution to the wave equation for the electric field is given:

$$E = a \cos(\beta (t c - z)) + b \cos(\beta (t c + z))$$

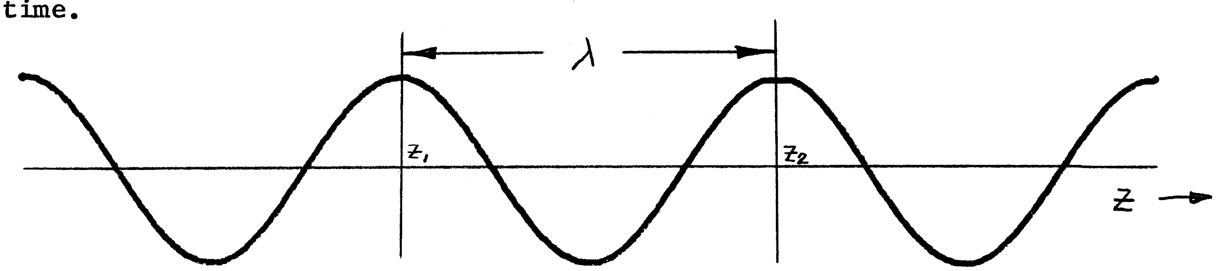
These are waves traveling in the plus and minus $z$ directions. $\beta$ is a constant which is equal to $2\pi$ divided by the wavelength. The wavelength




Propagating Wave in Time and Space


One Wavelength
Space
Sin(ωt - βz)
Propagating Wave in Time and Space
Time
T
One Period

is the distance at which the wave repeats itself. Consider a fixed time.




The cosine function repeats itself at $2\pi$ intervals.

$$\frac{2\pi}{\lambda}(t_o c - z_1) - \frac{2\pi}{\lambda}(t_o c - z_2) = 2\pi$$

$$\frac{2\pi}{\lambda}(z_2 - z_1) = 2\pi$$

$$z_2 - z_1 = \lambda$$

For a sinusoidal wave the reciprocal of the time between equal points of the cosine (sine) waves (the period) is equal to the frequency (cycles/sec. or hertz).

$$f = 1/T$$

Fix the space coordinate z and pick two times so that the difference of the two arguments of the cosine is equal to $2\pi$.

$$\frac{2\pi}{\lambda}(t_2 c - z) - \frac{2\pi}{\lambda}(t_1 c - z) = 2\pi$$

$$\frac{t_2 c}{\lambda} - \frac{t_1 c}{\lambda} = 1$$

$$t_2 - t_1 = T = \frac{\lambda}{c}$$

Therefore the frequency of the wave is given by:

$$f = c/\lambda$$

We define the radian frequency, $\omega$ to be equal to $2\pi f$. If we substitute this into sinusoidal waves we get the more familiar form of the wave.

$$E = a\cos(\omega t - \beta z) + b\cos(\omega t + \beta z)$$

ELECTROMAGNETIC WAVES

The diagram on the following page shows an electromagnetic plane wave propagating in the positive z direction. The electric field is aligned with the x axis. The magnetic field is aligned with the y axis. The electric field, the magnetic field, and the direction of propagation form a right hand system. Rolling the fingers of our right hand from the electric field on the X axis to the magnetic field on the Y axis, our thumb points in the direction of propagation. Note that the electric and magnetic fields are in phase.

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We can find the relationship between the magnitude of the electric and magnetic fields by using the differential Faraday's law.

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t}$$

There is only a $E_x$ component of the electric field and only a $H_y$ component of the magnetic field. If the curl E is expanded and only the Y component of the curl E is retained.

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$$

$$E_x = E_o \cos(\omega t - \beta z) \qquad H_y = H_o \cos(\omega t - \beta z)$$

$$\beta E_o \sin(\omega t - \beta z) = \mu H_o \omega \sin(\omega t - \beta z)$$

$$E_o = \frac{\mu H_o \omega}{\beta}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{\omega}{c}$$

$$E_o = \mu c H_o \qquad \text{but} \quad c = 1./\sqrt{\epsilon \mu}$$

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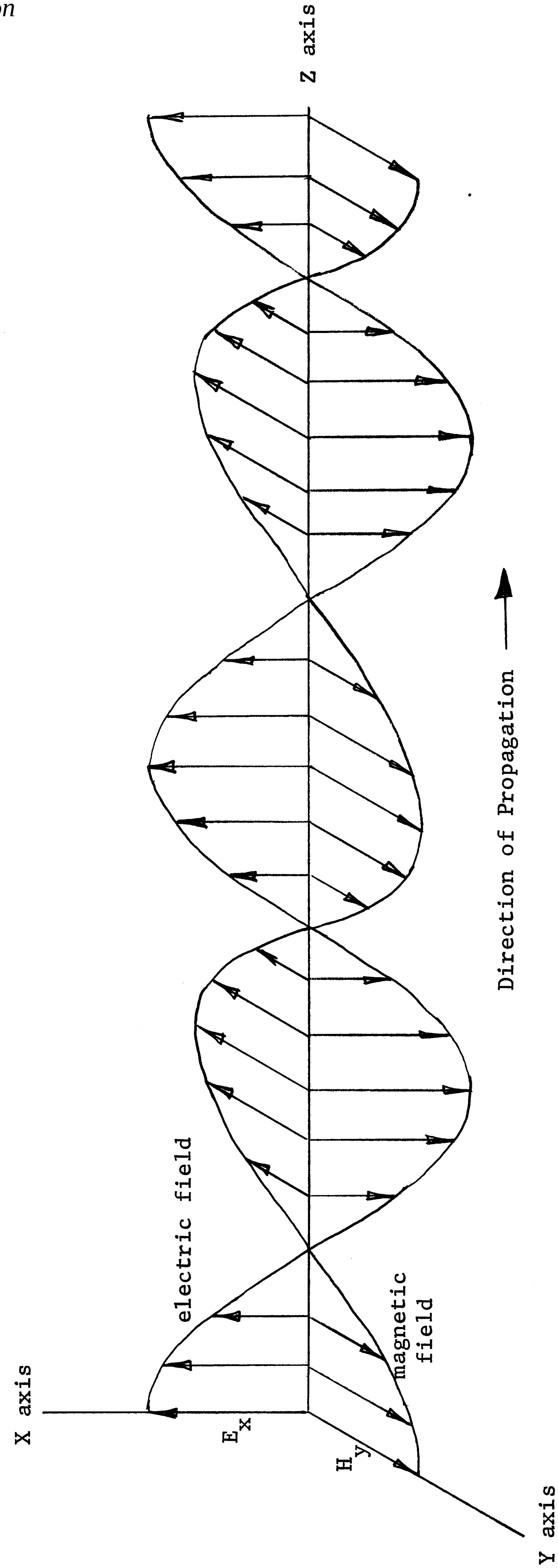
$$E_o = \sqrt{\frac{\mu}{\epsilon}} H_o$$

This is the equation which links the magnitudes of the electric and magnetic fields in an electromagnetic plane wave. The quantity $\sqrt{\mu/\epsilon}$ has the dimensions of ohms. This quantity is called the intrinsic impedance of the medium. In free space the intrinsic impedance Z is given as

$$Z = \sqrt{\frac{\mu}{\epsilon}} = 376.73 \text{ ohms}$$

If we know that we have a plane wave, which is the radiation from an antenna in the far field, then we only need to know the value of the electric or magnetic field and the other can be found. In antenna theory we usually only concern ourselves with the electric field.




Plane Electromagnetic Wave

## MATHEMATICAL REVIEW

### DECIBELS

In antenna measurements it is a lot of trouble to measure antennas in an absolute sense. It requires that the gain of the antenna be measured, which is an involved procedure requiring careful techniques to get accurate results. For most antenna design the ratio of pattern response in one direction to that in other directions is sufficient. The other problem is that the range of values of the antenna response is large. These problems are solved by using decibels. The decibel is given the symbol: dB. The dB is defined as the logarithm to the base 10 of the ratio of two powers, which is multipled by 10.

$$10 \, Log_{10} \left( \frac{P}{P_r} \right)$$

$P_r$ is the reference power. In a measurement it is only necessary to pick some point as the reference power level and all the measurements can be taken relative to that point. At some point it will be necessary to get an absolute measurement because the antenna may have great patterns (ratios of the response at different directions), but it may be inefficient. Many times we know that a structure is efficient from the materials it is made of and from transmission line measurements on the input connector, then we are not too concerned with measuring the efficiency. Note the logarithm has a subscript 10, this is to remind us that the Log is the common base and not the natural base (Ln). We will drop the subscript and reserve Log as the logarithm to the base 10.

The log of the ratio of two voltages can also be taken and expressed in dB. Remember that dB is defined as the ratio of powers. The voltage is related to the power as the square of voltage. If we want to take the ratio of two voltages in dB, then the voltages must be across the same impedance so that they can be related to power. The power is related to voltage as:

$$P = V^2/R$$

If we take the ratio of two powers across the same impedance, the impedance will cancel out and we are left with the ratio of two voltages squared.

$$\left( \frac{V}{V_r} \right)^2$$

Taking the Log of this ratio and remembering that the logarithm of the square of a number is equal to two time the logarithm of the number, the ratio of two voltages can be expressed.

$$20 \, Log \left( \frac{V}{V_r} \right)$$

One common reference power is milliwatt. In that case the expression dBm is used, the ratio of the power to a milliwatt. The other common reference power is the power that would be received by an antenna with an isotropic response. An isotropic antenna transmits power or receives power equally in all directions. This ratio of powers is expressed: dBi. The important thing to remember about dB is that it is a logarithm of a dimensionless quantity. It is a logarithm of the ratio of two quantities: power.

## PHASORS

Phasors are a method for representing a sinusoidal response or function by a complex NUMBER symbolism. In this course only sinusoidal forcing functions will be considered. That is, the input voltage will be expressed as:

$$V = V_0 \sin(\omega t)$$

Similiarly the electric field will be expressed in the same manner. The quantity $\omega$ is the radian frequency which is $2\pi F$, where F is the frequency in Hz (cycles/sec). This is not a restriction because any forcing function can be expressed in a Fourier transform or series and each sinusoidal component can be considered separately. We are seldom interested in the transient response of an antenna.

A quick review of complex number is in order. Any high school (college) algebra text covers complex numbers. It seems like an amusing exercise (it wasn't too amusing to some of us). Complex numbers are based on the square root of -1. In electrical engineering we call this square root: j. The mathematicians and physicists use i, but the engineers wanted to reserve i for the current. The two basic operations are given as:

$$(j)(j) = -1 \qquad (-j)(j) = 1$$

A complex number is expressed as having a real part and an imaginary part. This is a very bad choice of words because the imaginary part is just as real as the real part when using phasor notation.

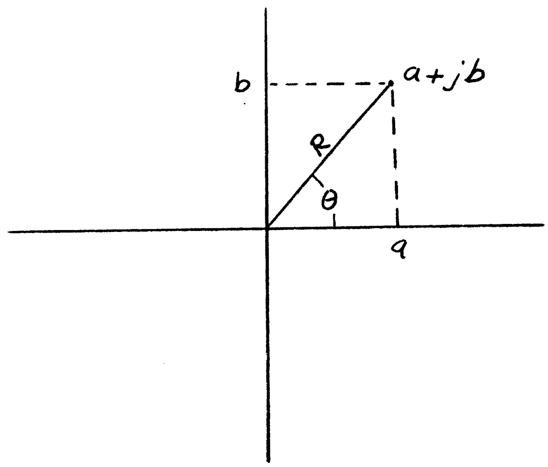
$$A = a + jb$$

This is a complex number. When adding or substracting two complex numbers, the real and imaginary parts are added separately. The multiplication and division require expansion of the result with the multiplication of the j terms carried out and the terms gathered together into real and imaginary parts. These operations, multiplication and division, are done easier using polar notation which is covered below.

$$(a + jb)(c + jd) = ac + jad + jbc + jjbd = (ac - bd) + j(ad + bc)$$

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{ac+bd+j(bc-ad)}{c^2+d^2}$$

The multiplication and division of two complex numbers is given above. The complex number can also be expressed in a polar notation. Take a two dimension coordinate system with the X axis as the real part of a complex number and the Y axis as the imaginary part. Then any complex number can be represented as a point in this two dimensional space. This point can also be represented as a distance from the origin and the angle from the X axis; the polar representation.




The polar representation uses the Euler's identity which defines the exponential function (the base of the natural logarithms (e)) with complex number arguments.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Using this, the polar representation is written.

$$a+jb = Re^{j\theta}$$

Where R is the magnitude of the complex number and $\theta$ is the phase angle. When the expression above is expanded using the Euler's identity, the complex number is recovered.

$$Re^{j\theta} = R\cos\theta + jR\sin\theta$$

$$a = R\cos\theta \qquad b = R\sin\theta$$

$$a^2+b^2 = R^2(\cos^2\theta + \sin^2\theta) = R^2$$

$$\frac{b}{a} = \frac{R\sin\theta}{R\cos\theta} = \tan\theta \longrightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Now we can consider the phasor concept. Given that we have a forcing function.

$$v(t) = V_m \cos(\omega t + \phi)$$

Using the Euler's identity we can say that our function is the real part of a complex representation. The real part is given by Re( ).

$$v(t) = Re\left(V_m e^{j(\omega t + \phi)}\right)$$

Remember that the exponential function has the following property:

$$e^{(a+b)} = e^a e^b$$

Then we can write the complex representation of the sinusoidal function as:

$$e^{j(\omega t + \phi)} = e^{j\omega t} e^{j\phi}$$

To make this a phasor, we let the factor $e^{j\omega t}$ be understood and also drop the real part function. We have expressed the sinsusoidal function in a simpler fashion. Of course, to get the right function we have to add back the $e^{j\omega t}$ term and take the real part. The function in phasor notation is:

$$V = V_m e^{j\phi}$$

The function is now expressed as magnitude and angle and is now called a phasor. We must keep track of the frequency. This saves us the trouble of writing out the sinusoidal function all the time. From here on all sinusoidal functions will be written as phasors, except in a few cases where it will be obvious.

## IMPEDANCE

Impedance is a good example of the use of phasors. Consider the equation for an inductor.

$$v(t) = L\frac{di}{dt}$$

Where L is the inductance and di/dt is the time derivative of the current. In other words the voltage across the inductor is equal to the inductance times the rate of change of the current in the inductor. If we assume as sinusoidal forcing function, then the equation can be transformed to phasor notation.

$$V e^{j(\omega t)} = L\frac{d}{dt}\left(I e^{j(\omega t)}\right)$$

The derivative of the exponential function is given by:

$$\frac{d}{dt}\left(e^{at}\right) = a e^{at}$$

Taking the time derivative, the result is:

$$V e^{j(\omega t)} = j\omega L e^{j(\omega t)} I$$

Now if we suppress the $e^{j\omega t}$ term, the phasor equation is obtained.

$$V = j\omega L I = Z I$$

The equation can be reduced to an Ohm's law equation as is done. The term Z is called impedance.

$$Z = j\omega L$$

Impedance is defined as the ratio of the phasor voltage to the phasor current. But the impedance itself cannot be transformed to the time domain by multiplying by $e^{j\omega t}$ and taking the real part. Impedance is a term only valid in the frequency domain. The frequency must be specified with the impedance value.

PHASOR OPERATIONS

We will now consider mathematical operations with phasors. Each phasor has magnitude and phase.

$$V = V_m e^{j\phi}$$

Where $V_m$ is the magnitude and $\phi$ the phase. If two phasors are multiplied together, we have:

$$V_1 e^{j\phi_1} \; V_2 e^{j\phi_2}$$

This is equal to:

$$V_1 V_2 e^{j\phi_1} e^{j\phi_2}$$

Above we saw that the exponential function has the following property.

$$e^{j\phi_1} e^{j\phi_2} = e^{j(\phi_1 + \phi_2)}$$

The resulting phasor has a magnitude of $V_1 V_2$ and a phase angle of $\phi_1 + \phi_2$. When phasors are multiplied, the magnitude of the result is the multiplication of the magnitudes; the resultant phase is the sum of the phase angles. Compare this with the multiplication of complex numbers expressed as: real part + j(imaginary part).

Phasor division is similiar to multiplication of phasors.

$$V_1 e^{j\phi_1} / V_2 e^{j\phi_2}$$

Again the magnitude of the result is the division of the two magnitudes. The exponential function has the following property.

$$\frac{1}{e^{j\phi_2}} = \left(e^{j\phi_2}\right)^{-1} = e^{-j\phi_2}$$

In general the exponential function has the property.

$$(e^a)^b = e^{ab}$$

The resultant phase angle of the division is given by:

$$e^{j\phi_1} e^{-j\phi_2} = e^{j(\phi_1 - \phi_2)}$$

The phase is the difference of the angles, $\phi_1 - \phi_2$. Compare this result with the above formula for division of two complex numbers.

When two phasors are added or substracted, each of them must be converted back to the real and imaginary parts. The the real parts are added together and the imaginary parts are added together.

$$V_1 e^{j\phi_1} + V_2 e^{j\phi_2}$$

$$= V_1 \cos\phi_1 + V_2 \cos\phi_2 + j(V_1 \sin\phi_1 + V_2 \sin\phi_2)$$

It is not too handy.

One of the concepts which has not been covered is the complex conjugate. If a complex number is given by:

$$V = a + jb$$

Then the complex conjugate is given as: $V^* = a - jb$

The sign of the imaginary part has been changed. The phase angle of the polar representation is changed when the complex conjugate operation is performed but the magnitude stays the same. The phase angle is related to the real and imaginary parts:

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$

The phase of the complex conjugate is given by: $\phi_c = \tan^{-1}\left(\frac{-b}{a}\right)$

From the property of the arctangent function, we find the result:

$$\phi_c = -\phi$$

Hence $$(Ve^{j\phi})^* = Ve^{-j\phi}$$

The magnitude stays the same but the phase angle changes sign. We can find the magnitude of the phasor by using the complex conjugate.

$$V^2 = (Ve^{j\phi})(Ve^{j\phi})^*$$

## RETARDED POTENTIALS

Consider the following thought experiment. Two charged particles are separated by a distance z. The particles each have a force applied to them due to the presence of the other charged particle. Now suppose one of the charged particles starts moving to another position farther away. What happens to the force between them? Does the stationary charge immediately have less force applied to it or does it take some time before the effect of the moving charge is felt? The answer is, of course, that the stationary charge does not know that the other charge has started to move until the message has had time to travel between them. The message is carried by an electromagnetic wave which travels with the speed of light. This is one of the consequences of the Theory of Special Relativity. The charge seen at a distance point is given by:

$$Q(t - z/c)$$

Where the charge is located a distance z away. The effect of a current is similiar. Consider a sinusoidal current at the origin, then the effective current at a distance r is given:

$$I = I_0 \cos \omega (t - \frac{r}{c})$$

This is called a retarded potential. The particular choice of t - r/c comes from our choice in the solution of the wave equation. Convert the current to a phasor and we have:

$$I = I_0 e^{-j\beta r} \qquad \beta = \frac{2\pi}{\lambda}$$

In standard engineering practice the phase angle decreases with increasing distance. We could have just as easily picked:

$$I = I_0 \cos \omega (\frac{r}{c} - t)$$

In fact, this is the choice that the physicists make.

We should consider the moving charged particle in the experiment above. What does it see when it starts to move? The answer is less force because the stationary charge has set up an electric field z/c seconds before which the moving charged particle immediately sees. It appears that the forces are no longer equal on the two different charges.

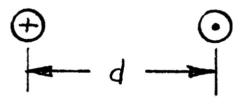
The retarded potential is the reason antennas radiate. A moving charged particle can not know until time 2 z/c later that there is another charged particle out there to receive the wave. So it must radiate without knowing there is a receiver of the message, because it could have moved, too, or been annihilated by another charge at the time the moving charge radiates the message. Because of the retardation time, an antenna radiates without the feed back that there is another antenna which will receive the power and load the wave. In a steady state when the antenna has been radiating for a while, the radiating antenna has its input impedance changed by the presence of the receiving antenna. For most situations the effect is too small to measure.

## ANTENNA RADIATION

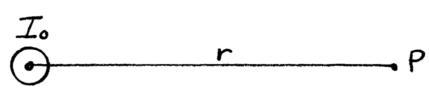
The purpose of this section is explain why an antenna radiates. In some sense this problem is the discussion of the whole course. None the less, it would be nice to have some feel for the radiation of antennas. Too many times it is so buried in mathematics that the physical concept is lost.

We discussed that charges cause electric fields and currents have magnetic fields. If the currents and charges are time varying, then it follows that the fields associated with them are also time varying. Maxwell showed that time varying fields are interrelated. We can say that either time varying field generates the other. The combination of time varying electric and magnetic fields propagates through space as an electromagnetic wave which we call radiation. All of this leads us to the opposite conclusion: why do not all circuits with time varying currents and charges radiate? In fact they do. But it is a matter of degree. An antenna is a circuit which is designed to radiate most of the energy fed to it. When a circuit starts radiating, the designer has big problems because strange unexpected results occur.

We will consider first a two wire transmission line carrying some frequency f. The current is expressed as a phasor. These lines are separated by a distance, d, as shown in the figure. The current on the right wire




is flowing out of the paper and the left wire has the current flowing into the paper. These are the currents for the instant of time shown; these currents will alternate these directions with time, of course. Let us just look at one of these wires and ignore the other for a moment.

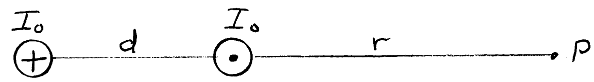



The fields at a point, P, can be calculated from the current on the wire. At this point it is not important how the fields are calculated. We can say that the field at point P is related to:

$$I_o e^{-j\beta r}/r \qquad \beta = \frac{2\pi}{\lambda}$$

Which is a retarded current in phasor notation. Suppose we would rotate the point P around the wire in a circular arc. Anywhere along this circular arc, the fields should be the same because of the symmetry. We will call this the pattern of the single wire radiating. Always take advantage of symmetry to get patterns. The effect of the current falls off by 1/r , the reciprocal of distance, which we will see later from conservation of energy considerations of a radiated field.

Now add back the second wire.




The fields at the point P are related to both of these terms:

$$I_o e^{-j\beta r}/r \qquad \frac{I_o e^{-j(\beta(r+d)+\pi)}}{r+d}$$

From symmetry conditions the two potentials from the currents will produce fields in the same directions. Note that the second potential has an $e^{-j\pi}$ term which is there because the current is in the opposite direction on the seond wire.

SUPERPOSITION

A material is called linear if its properties do not depend on the value of the fields. In that case the result of two fields impressed on a material is the sum of the fields. An antenna is linear because the properties of the materials used to make it. Fields are vector quantities and must be added as vectors. To add vectors, express both vectors using the same basis vectors and then the components may be added.

Back to the problem. The two potentials of current can be added because the fields caused by them will be in the same vector directions and free space is a linear material. We add the potentials and evalute $e^{-j\pi}$ as -1. The potential is given at point P.

$$I_o e^{-j\beta r}\left(\frac{1}{r} - \frac{e^{-j\beta d}}{r+d}\right)$$

For radiated fields we consider the point P to be far away. In this case we approximate $r + d \simeq r$. We will substitute r for r + d.

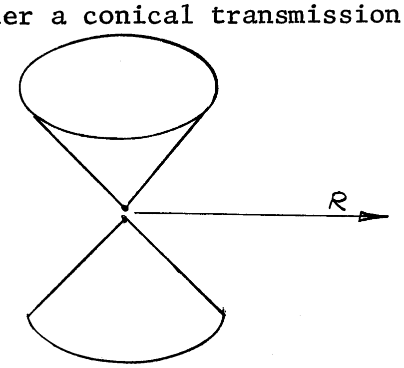
$$\frac{I_o e^{-j\beta r}}{r}\left(1 - e^{-j\beta d}\right)$$

This is the value of the radiation potential far away along the axis between the wires. As $\beta d \rightarrow 0$, the potential $\rightarrow 0$. The radiation potential at the point P increases as the distance in wavelengths between the wires increases. It reaches a maximum when $d = \lambda/2$. For low frequency circuits the return path of a current is close in wavelengths and the combination does not radiate significantly.

Show as an exercise that if the two wires are spaced on the X axis, then the radiation potential is zero along the Y axis.

A two wire transmission line will radiate energy unless the two wires are close together in wavelengths. Later we will consider the radiation from wires carrying current. Note that the only reason the antenna above radiates is because it obtained some size in wavelengths and because the retarded potential was used.

The second explanation of antenna radiation was formulated by Schelkunoff. In some respects it is my favorite because it has nice physical insight. Consider a conical transmission line as shown in the figure.




The characteristic impedance between the cones is a constant. If the line is fed from the center, the energy will spread out along the transmission line. The electric field is spherical between the cones. The wave will reach the end of the cone and be reflected by the open circuit. But the part of the field along the R axis direction cannot know until sometime later (retarded potential) that the transmission line was open circuited. The distance between the cones has grown to be a significant part of a wavelength. Also the circumferential distance has grown to be a significant part of a wavelength. We can say that the energy just flies off into space because it forgot it was part of a transmission line mode. That is not very rigorous but it has a nice physical feel to it. More exactly we can say that the energy reflected by the open circuit boundary is out of phase with the field reflected by the ends because of the retardation time finding out about the open circuit. Since this is out of phase, the total reflected field is reduced. Because the reflected energy is reduced, the difference is radiated by the antenna.

There are many antennas which can be thought of as diverging transmission lines. Examples are the biconical horn, as above, a waveguide horn, or even a dipole.