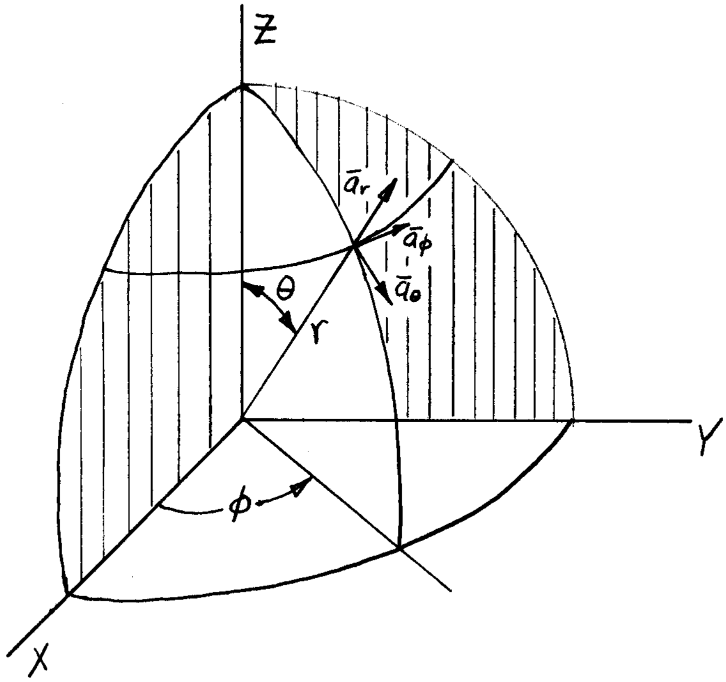
SPHERICAL COORDINATES



The spherical coordinates are used extensively in antenna design. The above diagram shows the spherical coordinates overlayed on the rectangular (X, Y, Z) coordinates. The first coordinate is the radial distance from the origin, R. The surface R = constant is a sphere. The second coordinate is the angle between the Z axis of the rectangular coordinates and the radius to the point. This angle is designated θ (theta). The surface θ = constant is a cone with its vertex at the origin. The third coordinate is also an angle. Project the radius line on to the X-Y plane. The angle it makes with the x axis is the angle ϕ (phi). The surface ϕ = constant is a plane which contains the Z axis.

The following is the set of relations between the rectangular and spherical coordinates.

$$R^2 = X^2 + Y^2 + Z^2 \qquad \cos\theta = \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}} \qquad \tan\phi = \frac{Y}{X}$$

$$X = R\sin\theta\cos\phi \qquad Y = R\sin\theta\sin\phi \qquad Z = R\cos\theta$$

Shown on the figure above are the directions of the unit vectors in the R, θ, ϕ directions. Each one of these unit vectors is in the direction of increasing coordinate. Many times the fields of antennas are expressed in terms of the theta and phi components.

TRANSFORMATION BETWEEN VECTORS IN RECTANGULAR AND SPHERICAL COORDINATES

Given a vector in rectangular coordinates.

$$\overline{V} = U_x \bar{a}_x + U_y \bar{a}_y + U_z \bar{a}_z$$

The radial component of the vector in spherical coorcinates is the projection of the vector on to the unit radial vector.

$$U_r = \bar{a}_r \circ (U_r \bar{a}_x + U_y \bar{a}_y + U_z \bar{a}_z)$$

$$U_r = U_x \bar{a}_r \circ \bar{a}_x + U_y \bar{a}_r \circ \bar{a}_y + U_z \bar{a}_r \circ \bar{a}_z$$

To reduce the above expression we need the vector dot products of the unit vectors.

$$\bar{a}_r \circ \bar{a}_x = \sin\theta\cos\phi \qquad \bar{a}_\theta \circ \bar{a}_x = \cos\theta\cos\phi \qquad \bar{a}_\phi \circ \bar{a}_x = -\sin\phi$$

$$\bar{a}_r \circ \bar{a}_y = \sin\theta\sin\phi \qquad \bar{a}_\theta \circ \bar{a}_y = \cos\theta\sin\phi \qquad \bar{a}_\phi \circ \bar{a}_y = \cos\phi$$

$$\bar{a}_r \cdot \bar{a}_z = \cos\theta \qquad \bar{a}_\theta \circ \bar{a}_z = -\sin\theta \qquad \bar{a}_\phi \circ \bar{a}_z = 0$$

The θ (theta) and ϕ (phi) components of the vector are found in a similiar manner.

$$U_\theta = U_x \bar{a}_\theta \circ \bar{a}_x + U_y \bar{a}_\theta \circ \bar{a}_y + U_z \bar{a}_\theta \circ \bar{a}_z$$

$$U_\phi = U_x \bar{a}_\phi \circ \bar{a}_x + U_y \bar{a}_\phi \circ \bar{a}_y + U_z \bar{a}_\phi \circ \bar{a}_z$$

If we expand the expressions above, we get the transformation from a vector in rectangular coordinates to a vector in spherical coordinates.

$$U_r = U_x \sin\theta\cos\phi + U_y \sin\theta\sin\phi + U_z \cos\theta$$

$$U_\theta = U_x \cos\theta\cos\phi + U_y \cos\theta\sin\phi - U_z \sin\theta$$

$$U_\phi = -U_x \sin\phi + U_y \cos\phi$$

The transformation from spherical coordinate vectors to rectangular coordinate vectors is found in the same way. The result is given below.

$$U_x = U_r \sin\theta\cos\phi + U_\theta \cos\theta\cos\phi - U_\phi \sin\phi$$

$$U_y = U_r \sin\theta\sin\phi + U_\theta \cos\theta\sin\phi + U_\phi \cos\phi$$

$$U_z = U_r \cos\theta - U_\theta \sin\theta$$

In general to transform from a vector in one coordinate system to a vector in another coordinate system, we need to find the projection (vector dot product) of the vector on to each of the unit vectors in the new coordinate system. The operation requires the knowledge of the vector dot products between all combinations of the unit vectors in the old coordinate system and the unit vectors in the new coordinate system. The set of unit vectors in each coordinate system is called a bases.

We will have to know the differential lengths of the three vectors in spherical coordinates. The product of these three differential lengths gives the differential volume in spherical coordinates. The three differential lengths are given:

$$dL_1 = dr \qquad dL_2 = r\,d\theta \qquad dL_3 = r\sin\theta\,d\phi$$

The most used differential area in spherical coordinate is the product of the θ and ϕ differential lengths. It is important to note that the differential volume and differential areas are not constant, but depend on the coordinates.

GENERALIZED CURVILINEAR COORDINATES

This is a good point to introduce generalized curvilinear coordinates. We can define a coordinate system, in which, given a point, the location is defined by three orthogonal (mutually perpendicular) surfaces at the point. The three surfaces are designated:

$$U_1 = \text{constant} \qquad U_2 = \text{constant} \qquad U_3 = \text{constant}$$

The three unit vectors are in the directions of increasing variables and are the normal vectors defining a tangent plane of the three surfaces defining the point.

$$\bar{a}_1\ ,\ \bar{a}_2\ ,\ \bar{a}_3$$

The differential lengths are given:

$$dL_1 = h_1\,dU_1 \qquad dL_2 = h_2\,dU_2 \qquad dL_3 = h_3\,dU_3$$

The differential volume is: $h_1\,h_2\,h_3\,dU_1\,dU_2\,dU_3$

The divergence of a vector $\bar{A}$ is:

$$\nabla\cdot\bar{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right)$$

Note the cyclic ordering of 1, 2, 3 in the expression.

The gradient of a scalar function V is:

$$\nabla V = \frac{\partial V}{h_1 \partial u_1}\bar{a}_1 + \frac{\partial V}{h_2 \partial u_2}\bar{a}_2 + \frac{\partial V}{h_3 \partial u_3}\bar{a}_3$$

The curl of a vector is given as:

$$\nabla \times \bar{A} = \begin{vmatrix} \frac{\bar{a}_1}{h_2 h_3} & \frac{\bar{a}_2}{h_3 h_1} & \frac{\bar{a}_3}{h_1 h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

The Laplacian of a scalar function V is:

$$\nabla^2 V = \nabla \cdot \nabla V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right]$$

If two vectors are defined at the same point in the general curvilinear coordinate system, then the following two expressions are true:

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

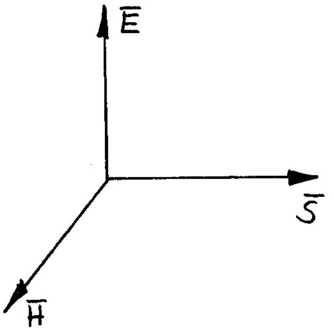
$$\bar{A} \cdot \bar{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

One of the problems that occurs in antenna theory is the vector dot product of a radius vector to a far field point which is defined from the origin and a source point not at the origin. The only proper way to take the vector dot product in spherical coordinates is to express the radius vector in terms of the unit vectors at the source point. Most of the time we just use the rectangular coordinates which do not change directions of unit vectors for different points.

Common coordinates:		1	2	3
Rectangular	U_i	X	Y	Z
	h_i	1	1	1
Cylindrical	U_i	R	ϕ	Z
	h_i	1	R	1
Spherical	U_i	R	θ	ϕ
	h_i	1	R	R $\sin\theta$

POYNTING VECTOR

The electric and magnetic fields are perpendicular in an electromagnetic wave. Both fields are perpendicular to the direction of propagation. If we take the vector cross product of the two fields, we will get a vector in the direction of propagation.



$$\bar{S} = \frac{1}{2}\,\bar{E} \times \bar{H}^*$$

Where E and H are the peak values of the phasors. If we use RMS values, then $\bar{S} = \bar{E} \times \bar{H}$. The dimensions of the Poynting vector are watts/(meter)^2. It is the energy density of the electromagnetic wave.

The magnitude of the Poynting vector is the product of the magnitude of the components in the plane wave.

$$|\bar{S}| = \frac{1}{2}|E||H^*|$$

We can express the magnetic field magnitude as a function of the electric field in a plane wave in free space.

$$|\bar{H}| = \frac{|\bar{E}|}{\eta} \qquad \eta = 376.73$$

The Poynting vector can then be expressed in terms of the electric field only.

$$|\bar{S}| = \frac{1}{2}\frac{|\bar{E}|^2}{\eta}$$ E is the peak value.

$$|\bar{S}| = \frac{|\bar{E}|^2}{\eta}$$ E is the RMS value.

SPHERICAL WAVES

We have only talked about plane waves. An actual antenna radiates spherical waves, but at large distances the spherical waves can be approximated by plane waves. We will use plane waves again because they simplify the analysis. The problem with plane waves is that they require infinite energy, not so with spherical waves. In spherical waves the propagation is in the R direction. Since R is the direction of propagation, the Poynting vector is in the $\bar{a}_r$ direction. The energy density is distributed around a sphere. Consider a sphere centered on the antenna with radius R. Assume R is great enough so that the near by induction fields (non propagating) have become insignificant (far field). At this point we can find the total power radiated by integrating the magnitude of the Poynting vector over the surface of the sphere. The differential area on the sphere is given as $R^2 \sin\theta \, d\phi \, d\theta$, which is $dL_2 \, dL_3$ of spherical coordinates. If we sum the magnitude of the Poynting vector times these differential areas over the surface of the sphere, we get the total power radiated.

$$P_r = \int_0^{2\pi}\int_0^{\pi} S_r R^2 \sin\theta \, d\theta \, d\phi$$

The total power radiated is also the average value of the magnitude of the Poynting vector times the area of the sphere.

$$P_r = S_{r_{AV}} (4\pi R^2)$$

Suppose we look at two concentric spheres centered on the antenna, then the average Poynting vector magnitudes on the two spheres are related:

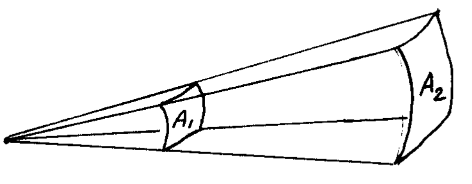
$$S_{1_{AV}} 4\pi R_1^2 = S_{2_{AV}} 4\pi R_2^2$$

The ratio of the average Poynting vector magnitude is:

$$\frac{S_1}{S_2} = \frac{R_2^2}{R_1^2}$$

The average Poynting vector magnitude is proportional to $1/R^2$.

Let us consider a small area on the sphere. When the radius is doubled, the surface area grows by four. The Poynting vector only has an $\bar{a}_r$ component. The energy density function does not travel in the θ or ϕ



directions; therefore all the energy in the area A_1 is propagated to the area A_2. The sides of the figure above are called the bounds of a flux tube, the energy does not cross these boundarys. We will use the concept of flux tubes when we discuss ray tracing. Each ray contains a differential amount of energy which remains constant. Using the idea of flux tubes, we can shrink the area to the differential area and see that the Poynting vector magnitude at a radius R_1 is directly related to the magnitude at R_2. The relationship is the same as for the average Poynting vector magnitude. The Poynting vector magnitude at a particular θ and ϕ is proportional to $1/R^2$.

We saw above that the electric field magnitude is proportional to the square root of the Poynting vector magnitude. This means that the electric field magnitude is proportional to 1/R. Similiarly the magnetic field of a radiating spherical wave is proportional to 1/R. This is the reason the retarded potential used in the example of two wire radiation was proportional to 1/R. The magnetic (and electric) fields are directly proportional to the retarded potential of current.

RADIATION INTENSITY

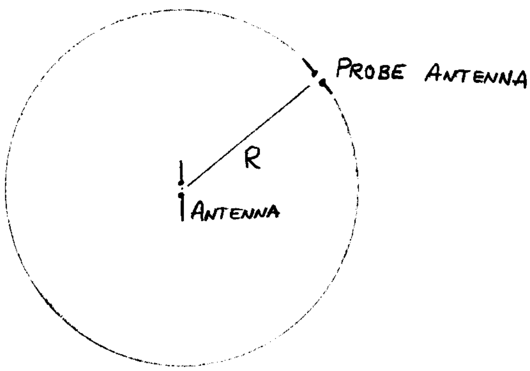
We saw that in a spherical wave, the magnitude of the Poynting vector was bound in flux tubes which are radial lines. It is convenient to define the radiation intensity which does not have the $1/R^2$ dependence.

$$U = S_r R^2 \qquad \text{(watts/ Solid angle)}$$

This is a function of θ and ϕ : $U(\theta, \phi)$. The antenna does not radiate equally in all directions which we call the pattern.

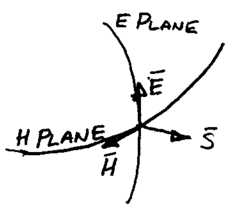
ANTENNA PATTERN

The pattern is a measure of the radiation intensity. To measure the pattern we must measure the power density at a fixed distance from the antenna. The power density is measured with a second antenna.



The probe antenna can either be carried around the antenna in a circle or the antenna in the center can be rotated. The first method is used with large antennas and the second with small antennas.

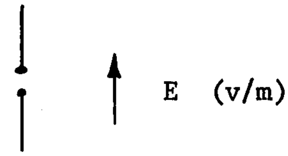
If we have a linearly polarized antenna, it has a spherical wave which has the electric field in only one direction. In that case there are two special patterns. The first is in the plane containing the electric field. This is called the E plane cut. The other special pattern is the H plane cut; it is in the plane containing the magnetic field.



We have named the patterns which cut through the constant surfaces of the spherical coordinate system angles. A pattern through a constant ϕ coordinate is called a great circle cut. A pattern through a constant θ coordinate is called a conical cut (the surface of constant θ is a cone).

INCREMENTAL DIPOLE PATTERN

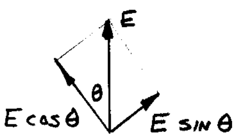
Assume a small dipole is located in a uniform field parallel to the arms of the dipole.



The electric field will induce charges on the arms of the dipole. Since the field varies as $E_o \sin\omega t$, the induced charges vary as $q(t) = B E_o \sin\omega t$ where B is some constant. If we now connect the center to a transmission line, the induced charges are drawn off on to the transmission line.

$$I = \frac{dq}{dt} = I_o \cos\omega t$$

Now let us rotate the antenna relative to the field. The field may be



divided into two components as is done in the figure above. The current induced in the transmission line is now: $I = I_o \cos\omega t \cos\theta$. The power received is $\frac{1}{2} R I^2$ where R is the impedance of the transmission line.

$$\frac{1}{2} R I^2 = \frac{R I_o^2}{2} \cos^2\theta$$

If we normalize the power received to the maximum, we have

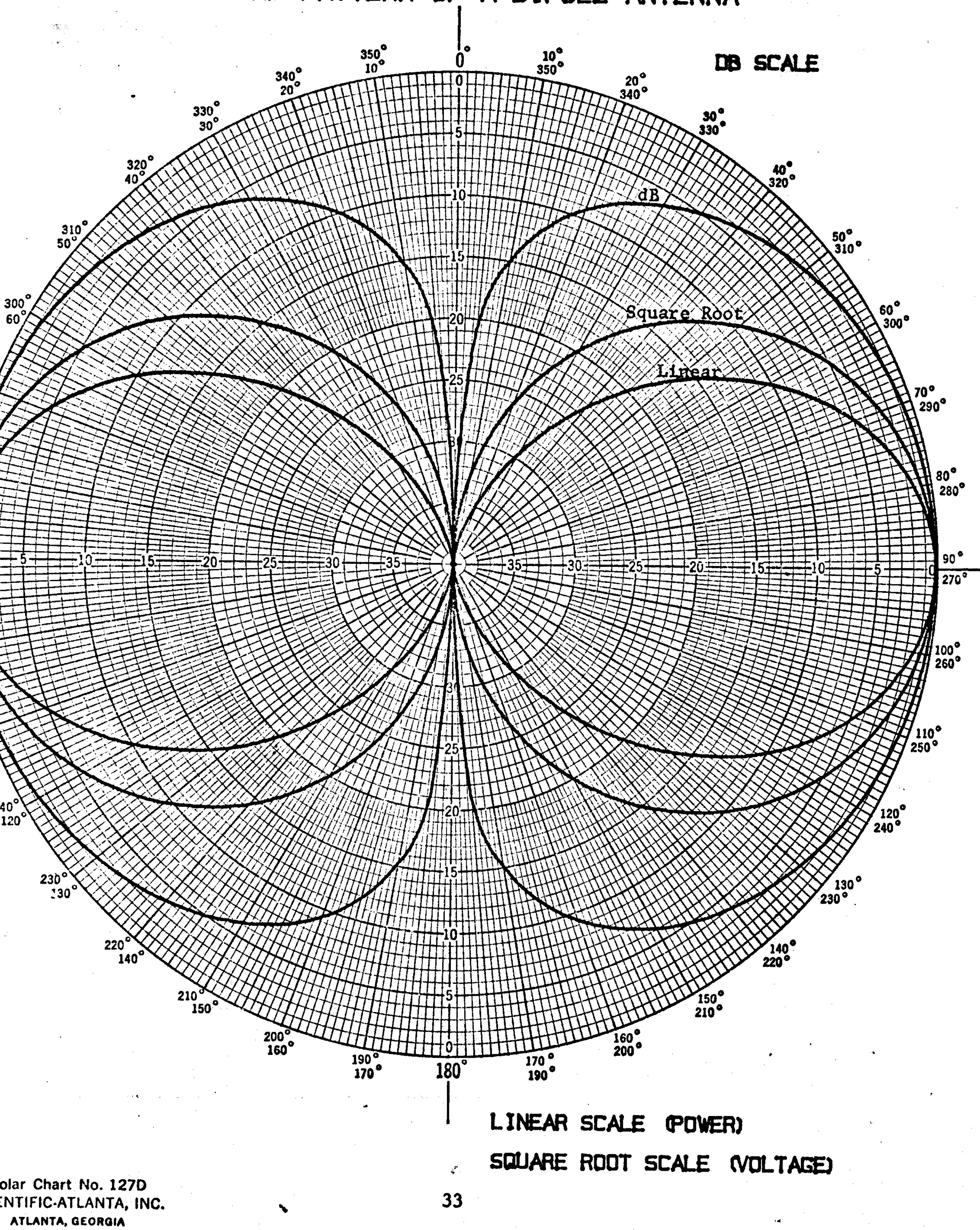
$$U = U_o \cos^2\theta$$

Which is the pattern of the increment dipole. On the following page is a polar pattern of the incremental dipole. It is plotted using the three types of pattern scales.

PATTERN SCALES

The polar pattern shows the three types of pattern scales. The outer pattern which is the most used one, is the decibel scale. The paper used is dB paper with the numbers of the scale in dB with the maximum power at the edge of the paper. The values toward the center are actually negative dB although they are given as positive numbers. It has the advantage that it can show a large variation of the pattern magnitude. The - 40 dB point corresponds to 1/10000 th the power. The sidelobes show up on a dB scale. The dB scale must be referenced to some power level, many times the maximum signal level. The second

E PLANE PATTERN OF A DIPOLE ANTENNA



pattern in is the square root pattern. The pattern should be plotted on linear paper with 100 on the outside circle and 0 at the center, but since the dB scale is the most used plot, it is plotted on top of it. This is the square root of the radiation intensity which is proportional to the electric field or magnetic field. The square root plot is sometimes referred to as the voltage plot. The third pattern scale is proportional to the radiation intensity or the power pattern. It is also called the linear pattern.

DIRECTIVITY

Directivity is a measure of the concentration of the radiation intensity in the direction of the maximum radiation intensity.

$$\text{Directivity} = \frac{\text{Maximum Radiation Intensity}}{\text{Average Radiation Intensity}} = \frac{U_{max}}{U_o}$$

The average radiation intensity, U_o, is the power radiated divided by the total spherical solid angle, 4π, which is the area of the unit sphere. The total power radiated is the surface integral of the power density.

$$\text{Average Radiation Intensity} = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} U(\theta,\phi)\sin\theta\, d\phi\, d\theta$$

The directivity is related to the gain of the antenna, but it is easier to measure or estimate from a few measurements. The gain equals the directivity times the efficiency. The integral formula for directivity is:

$$\text{Directivity} = \frac{4\pi U_{MAX}}{\int_0^{\pi}\int_0^{2\pi} U(\theta,\phi)\sin\theta\, d\phi\, d\theta}$$

The term directive directivity has been defined for an arbitrary direction as the radiation intensity divided by the average radiation intensity.

The directivity may be expressed in dB. The directivity is dimensionless and is the ratio of two powers, so the directivity in dB is 10 times the log to the base 10 of the ratio.

DIRECTIVITY ESTIMATES

The directivity may be found by measuring the radiation intensity relative to some convenient reference level at equal angle increments over the whole radiation sphere (16380 points for 2^o increments). All these values are integrated numerically to find the average radiation intensity and the above formula is used with the maximum measured radiation intensity. If the pattern is well behaved then the directivity can be estimated from one or two patterns. The usual patterns are the principle plane cuts which are the E plane and H plane patterns.

BEAMWIDTH

When a pattern has a single main lobe, we can define a beamwidth for the pattern. The most usual one is the half power beamwidth or the 3 dB beamwidth. We also use the 10 dB or tenth power beamwidth when working with parabolic reflector feed antennas. All the directivity estimates require a distinct beamwidth usually centered at $\theta = 0$.

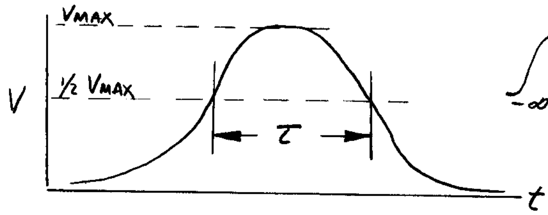
KRAUS METHOD

Given a pencil beam centered on $\theta = 0$, the directivity can be estimated from the beamwidths of the E plane and H plane patterns

$$\text{Directivity} \simeq \frac{41253}{\theta_E\, \theta_H} \qquad \text{(ratio)}$$

Where θ_E, θ_H are the beamwidths in degrees.

From circuit theory we have the following estimate of the power in a pulse.



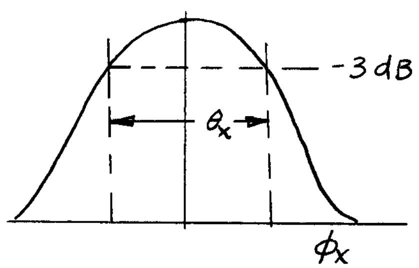
$$\int_{-\infty}^{\infty} V(t)\,dt \cong V_{MAX}\,\tau$$

A coordinate system that is often used with pencil beam antennas is the elevation-azimuth coordinate system, ϕ_x, ϕ_y.

ϕ_x is the angle from the Y - Z plane

ϕ_y is the angle from the X - Z plane

Consider the pattern plotted on ϕ_x.



$$\int_{-\pi}^{\pi} U(\phi_x)\, d\phi_x \cong 2 \sin \theta_x/2$$

Similiarly $$\int_{-\pi}^{\pi} U(\phi_y)\, d\phi_y \cong 2 \sin \theta_y/2$$

Then the directivity is given by: $$\text{Directivity} = \frac{4\pi}{4 \sin \theta_x/2 \, \sin \theta_y/2}$$

For small angles we can approximate the $\sin\theta$ by θ.

$$\text{Directivity} = \frac{4\pi}{\theta_E\, \theta_H}$$ Where the beamwidths are measured in radians.

Converting from radians to degrees we have

$$\text{Directivity} = \frac{41253}{\theta_E\,\theta_H}$$

Note that this is a ratio. There are other similiar formulas in the literature where other constants are used, but they are all about the same. The formula is better than it would appear from the number of approximations made.

CONICAL BEAM METHOD

This is a method of estimating the directivity where the pattern is approximated by an analytical function. The directivity of this pattern function can be calculated exactly and used as an estimate for the directivity. Consider a pattern given by the following formula.

$$U = \mathrm{Cos}^{2N}\left(\frac{\theta}{2}\right)$$

This is a convenient pattern function which we will use many times for theoretical pattern calculations. It is called the conical beam approximation. On page 37 four polar patterns are plotted using this function. The beam peak is at $\theta = 0$ and there is a null in the pattern at $\theta = 180^\circ$. The handy thing about this function is that the parameters of the pattern can be easily calculated. The 3 dB beamwidth of the conical beam pattern is:

$$\text{Beamwidth} = 4\ \mathrm{Cos}^{-1}\left(\left(\tfrac{1}{2}\right)^{1/2N}\right)$$

Given the 3 dB beamwidth, the constant N is found by:

$$N = \frac{\mathrm{Log}(\frac{1}{2})}{2\ \mathrm{Log}(\ \mathrm{Cos}(\mathrm{HPBW}/4))} \qquad \text{HPBW - Half Power Beamwidth}$$

The pattern can be integrated exactly. The directivity = N + 1. The conical beam approximation can be used for antennas with different E and H plane beamwidths if the following formula is used.

$$U(\theta, \phi) = \mathrm{Cos}^{2N_e}\left(\frac{\theta}{2}\right)\ \mathrm{Cos}^2(\phi) + \mathrm{Cos}^{2N_h}\left(\frac{\theta}{2}\right)\ \mathrm{Sin}^2(\phi)$$

This function can also be integrated and the directivity estimated. A nomograph has been drawn for this function giving the directivity from the E and H plane beamwidths and is given on pages 38 and 39. The same nomograph has been drawn twice with different scales.

CONICAL BEAM DIPOLE PATTERN

Many antennas will have a null in the pattern at 90° in the E plane mainly because the antenna is made from a combination of dipole elements. In these cases the conical beam approximation does not provide this null.

CONICAL BEAM ANTENNA PATTERN

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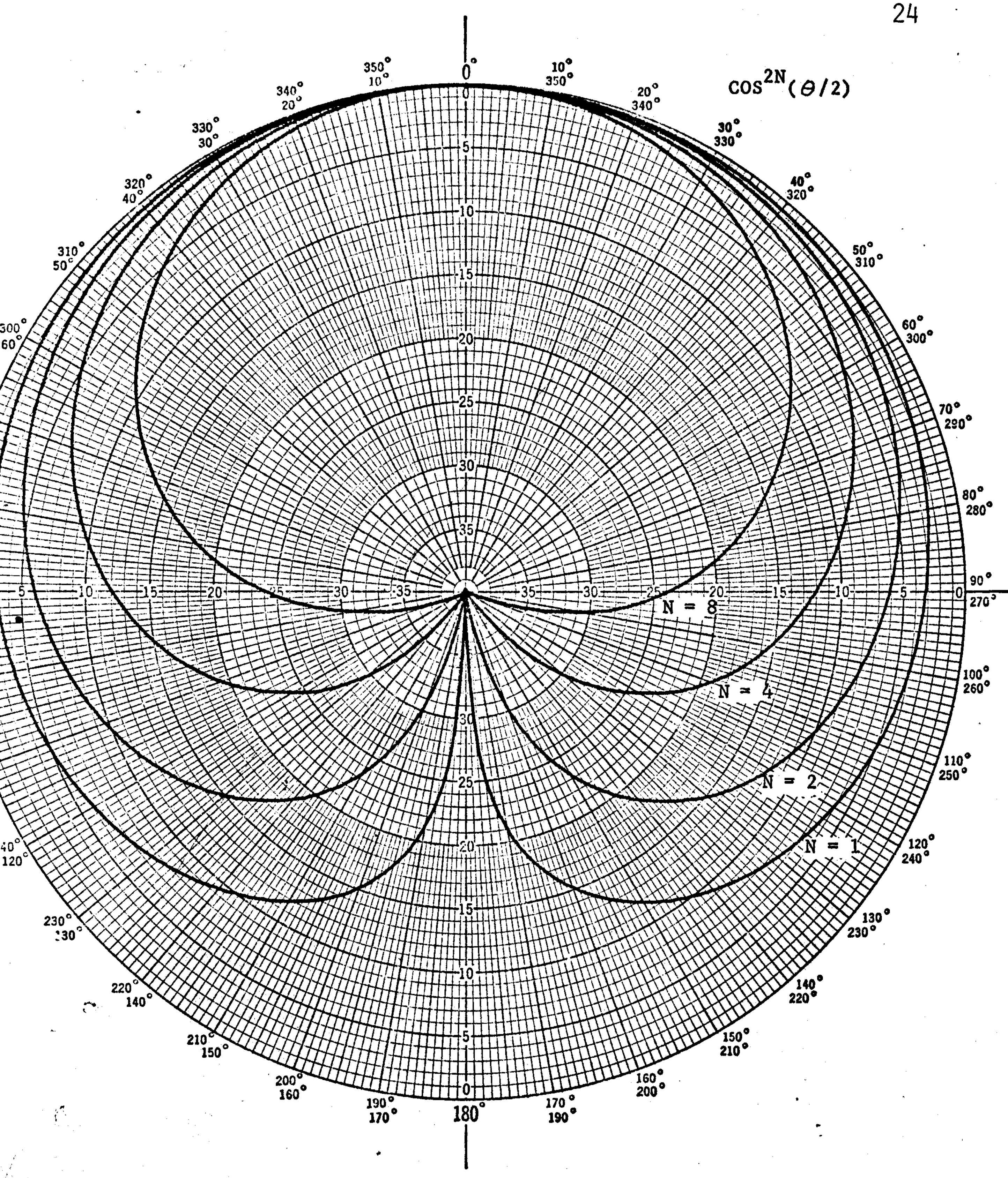

Polar Chart No. 127D
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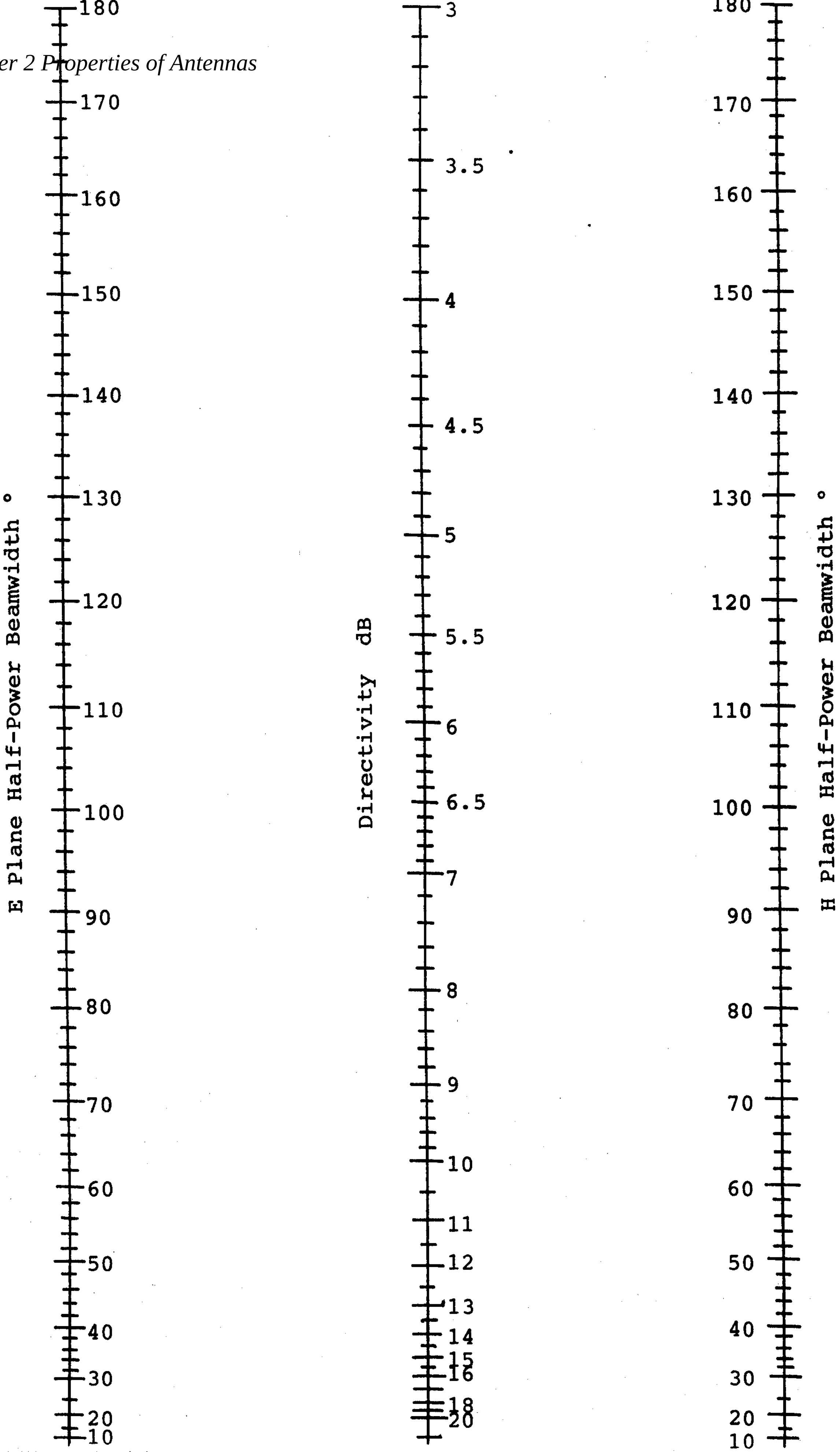
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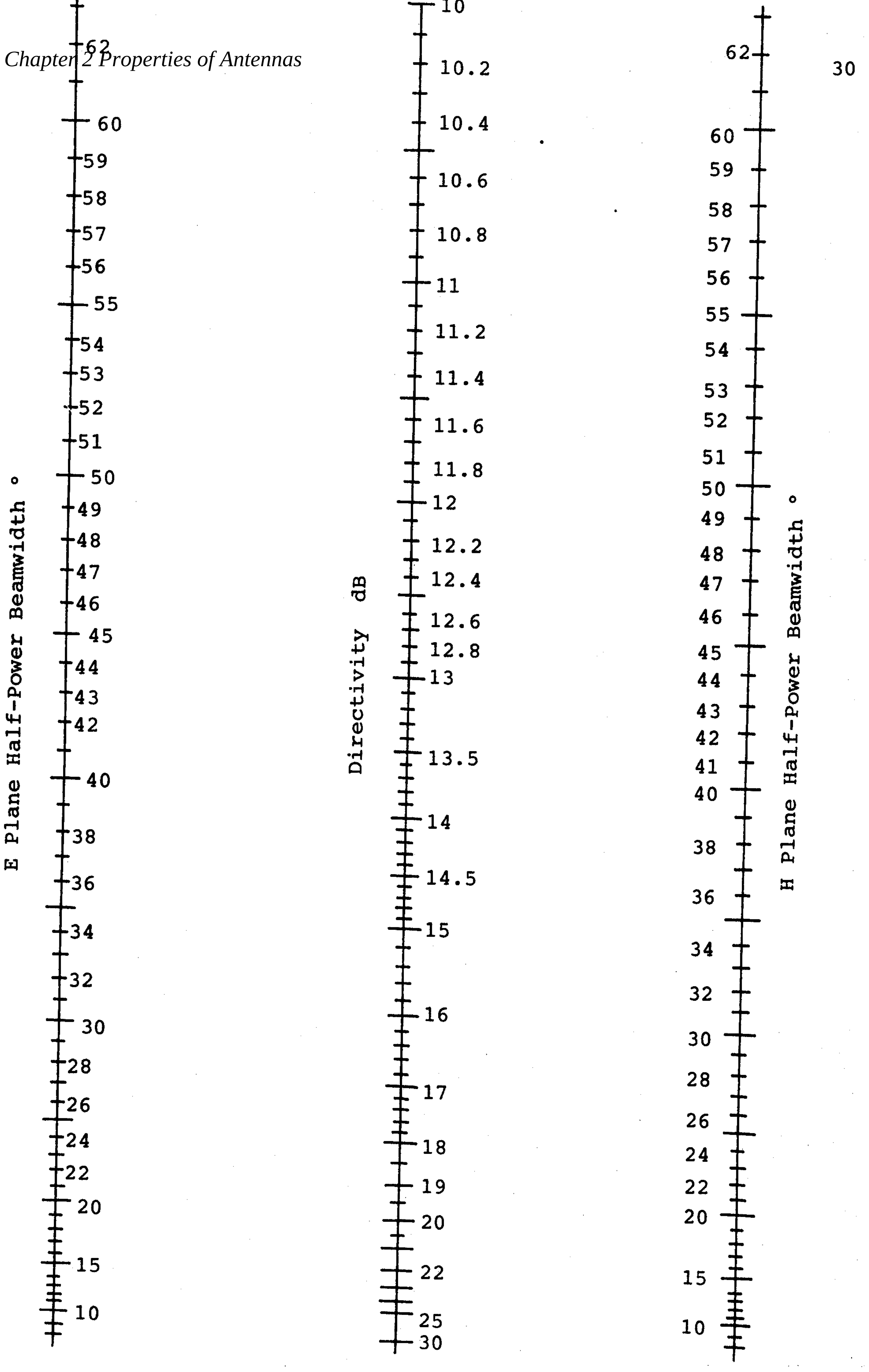
E Plane Half-Power Beamwidth °
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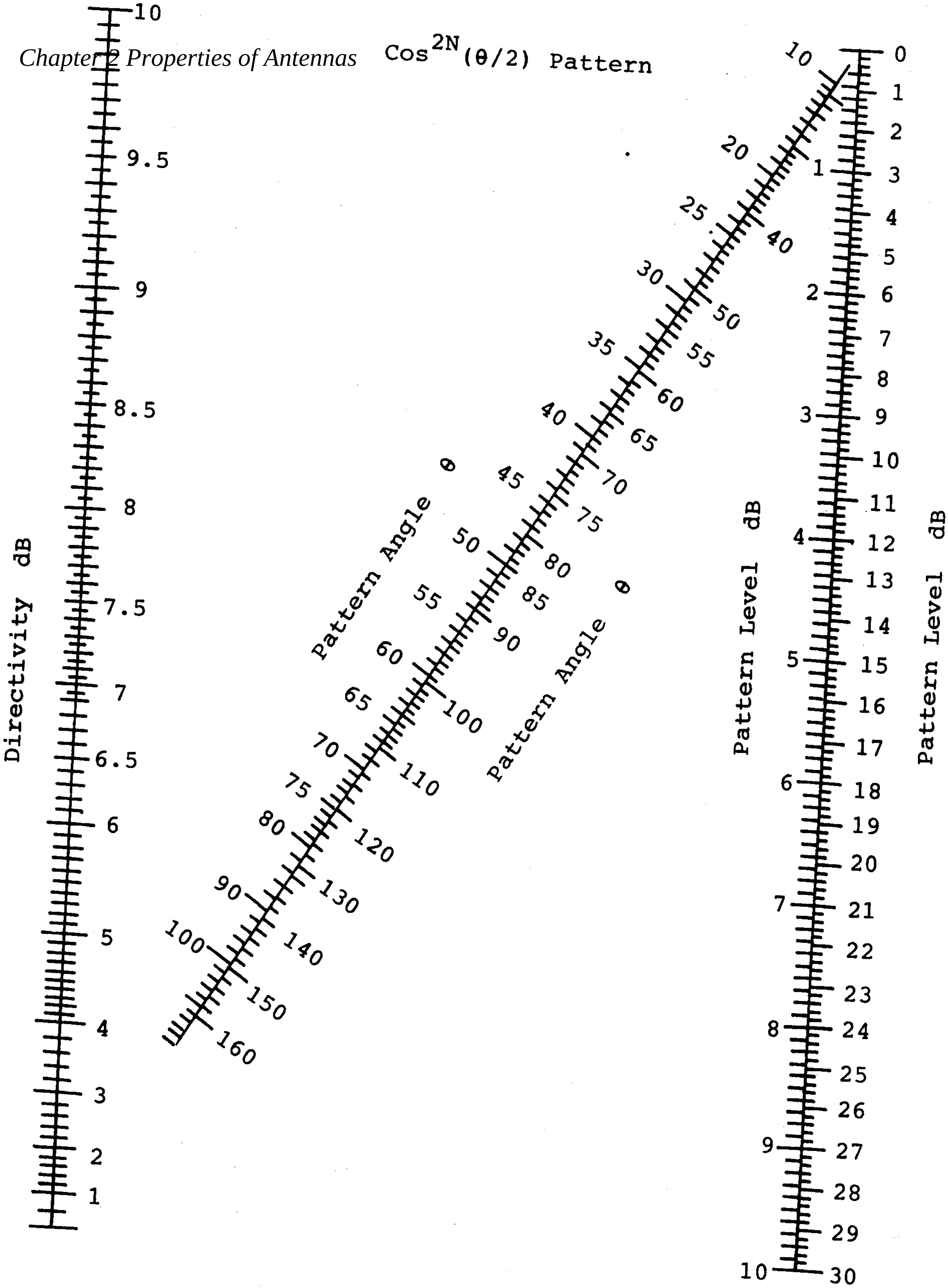
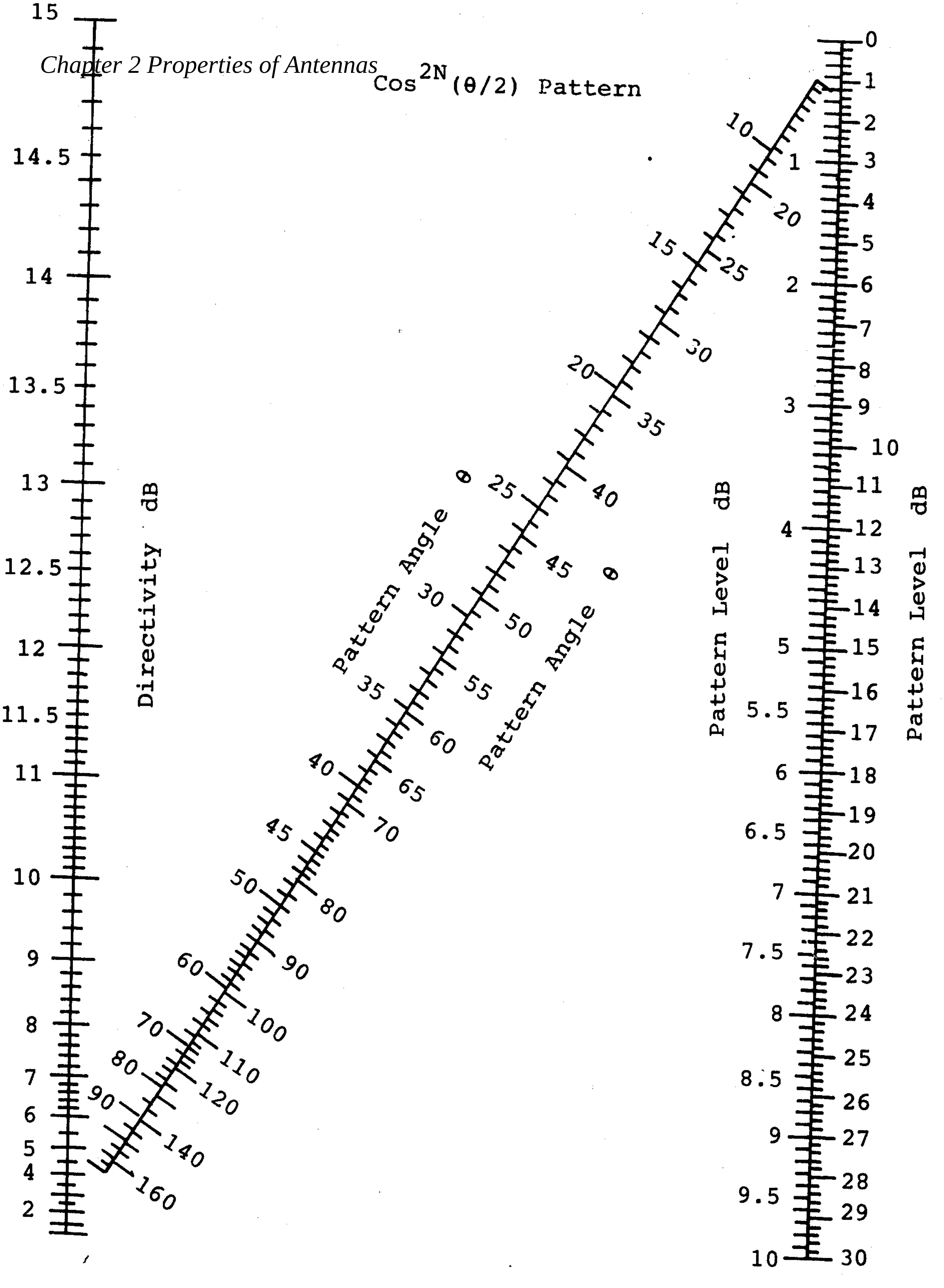
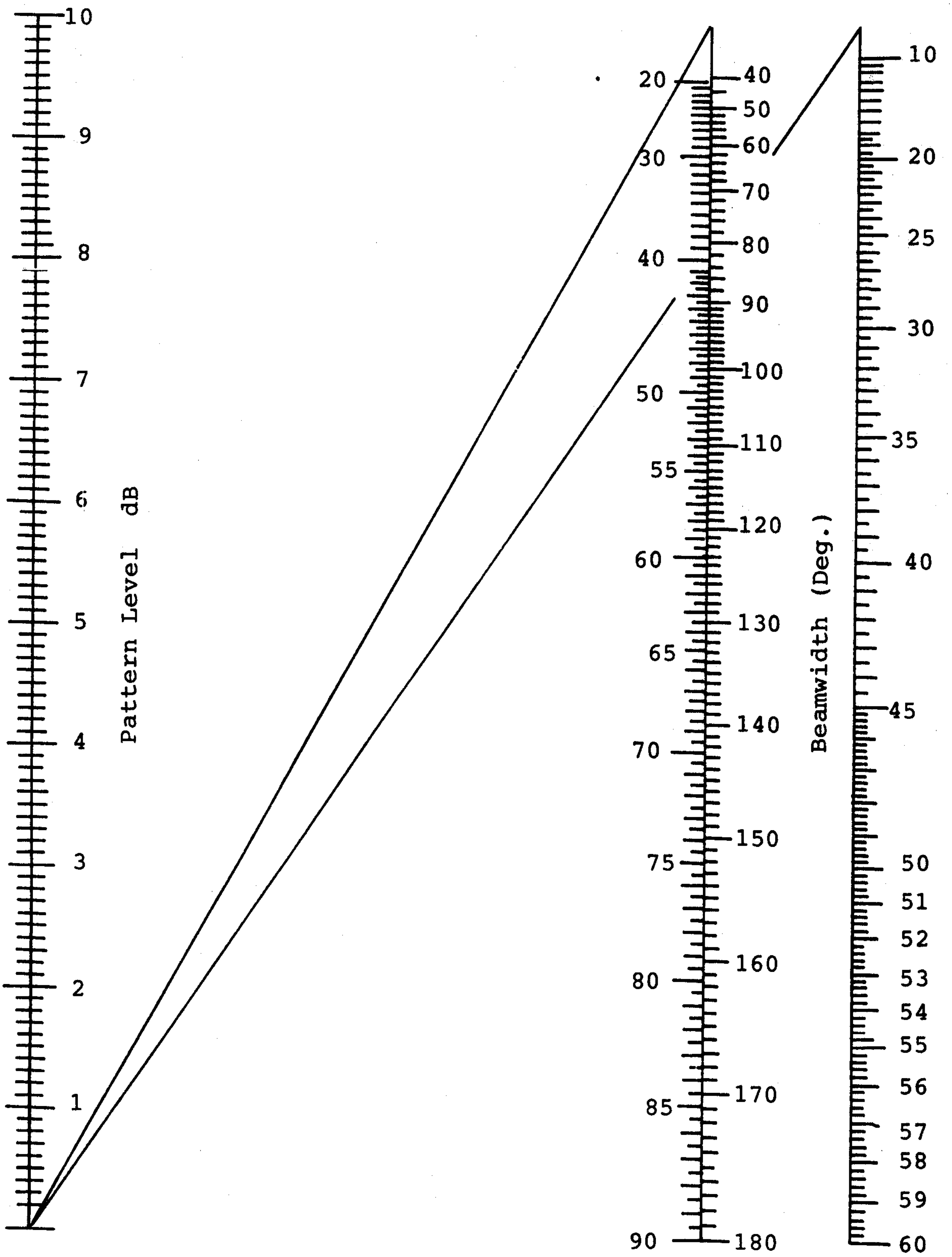
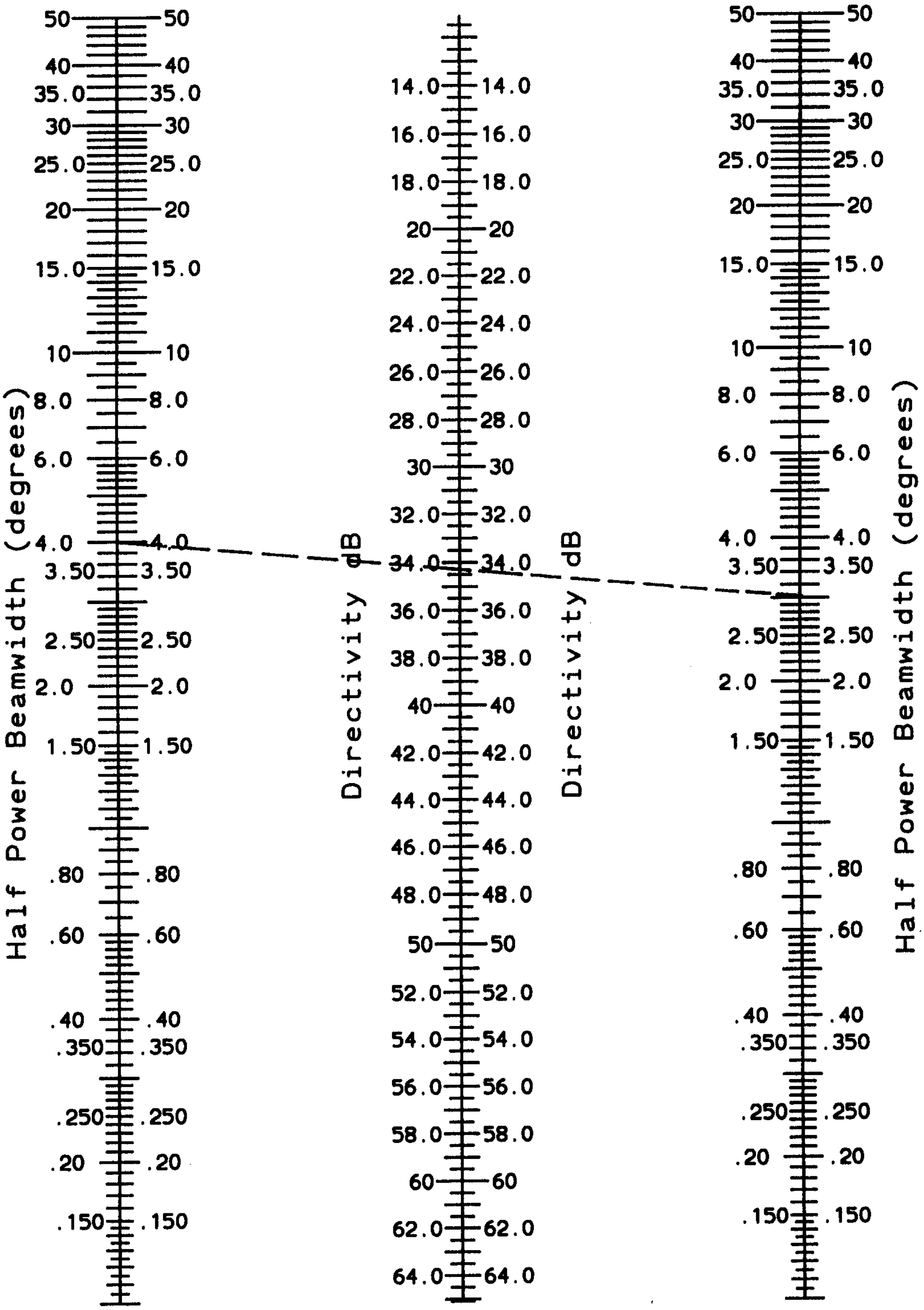
Cos^{2N}(θ/2) Pattern

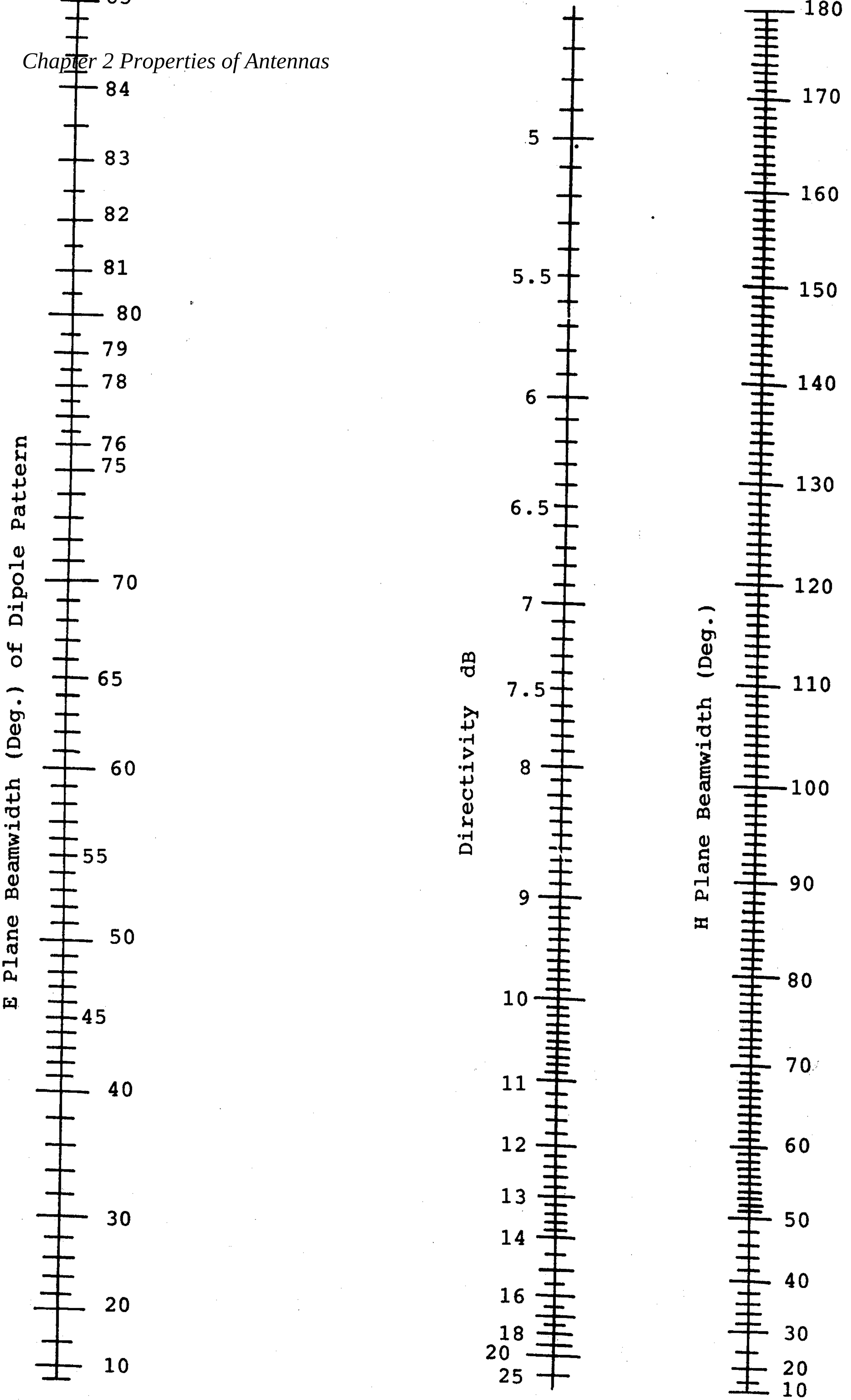



Cos^2N(θ/2) Pattern
Directivity dB
Pattern Angle θ
Pattern Angle θ
Pattern Level dB
Pattern Level dB



Approximate Relation of Beamwidths at Different Levels


Half Power Beamwidth (degrees)
50
40
35.0
30
25.0
20
15.0
10
8.0
6.0
4.0
3.50
2.50
2.0
1.50
.80
.60
.40
.350
.250
.20
.150
Directivity dB
14.0
16.0
18.0
20
22.0
24.0
26.0
28.0
30
32.0
34.0
36.0
38.0
40
42.0
44.0
46.0
48.0
50
52.0
54.0
56.0
58.0
60
62.0
64.0
Half Power Beamwidth (degrees)


E Plane Beamwidth (Deg.) of Dipole Pattern
84
83
82
81
80
79
78
76
75
70
65
60
55
50
45
40
30
20
10
Directivity dB
5
5.5
6
6.5
7
7.5
8
9
10
11
12
13
14
16
18
20
25
H Plane Beamwidth (Deg.)
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10

A pattern approximation that includes the conical beam and the dipole type pattern is given by:

$$U(\theta,\phi) = \cos^{2N_e}\left(\frac{\theta}{2}\right)\cos\theta\cos^2\phi + \cos^{2N_h}\left(\frac{\theta}{2}\right)\sin^2\phi$$

This function approximates well the pattern of a log periodic dipole antenna which has nulls at $\theta = 90^\circ$ in the E plane. The function can be integrated and an estimate for the directivity can be found. There is a nomograph on p. 41 using this approximation function to find the directivity from the E and H planes beamwidths. It is very similiar to the conical beam directivity chart.

For narrow beamwidths Kraus's formula gives the best results and for wide beamwidths the conical beam method gives the best results.

BUTTERFLY PATTERNS

There is a class of patterns for which the directivity estimates given above will not work. These patterns have a null on the Z axis and usually have rotational symmetry about the Z axis. A pattern with rotational symmetry about the Z axis will have identical great circle patterns for all ϕ. Patterns like this are generated by Mode 2 log periodic conical spirals, shaped reflectors, and some higher order waveguide horns. A pattern like this is sometimes called a butterfly pattern. The figure on page 42 is an example of the polar plot of a butterfly pattern (great circle cut). In general the pattern has main lobes which are centered at some angle θ (50 degrees in the figure). The beamwidth is measured about the beam center. The pattern is sometimes called a conical beam (confusing with the conical beam approx.) because the beam peak is at a constant θ.

A formula for the directivity similiar to the Kraus formula for pencil beams can be generated. In this case all the power in the pattern can be approximated by the width of the 3 dB beamwidth times the maximum power. Suppose the 3 dB down points are at angles θ_1 and θ_2. If we assume all the power is concentrated between these two angles and is of unit magnitude, we can integrate to find the average radiation intensity.

$$U_{AV} = \frac{1}{2}\int_{\theta_1}^{\theta_2} \sin\theta \, d\theta = \frac{\cos\theta_1 - \cos\theta_2}{2}$$

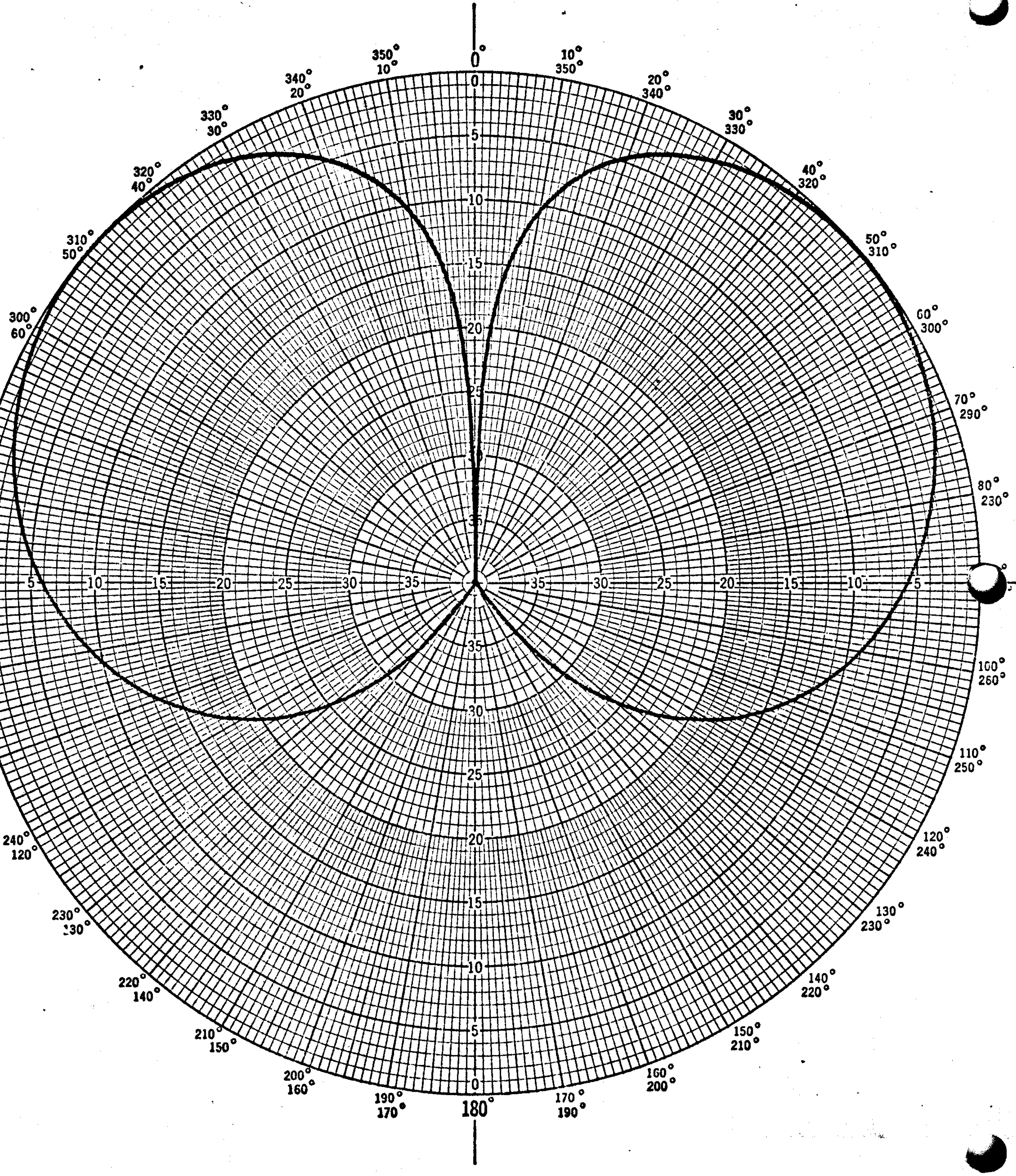
There is no need to integrate over the ϕ component because there is rotational symmetry. The maximum value of the radiation intensity is one; so we can find the directivity.

$$\text{Directivity} = \frac{2}{\cos\theta_1 - \cos\theta_2} \qquad \text{(ratio)}$$

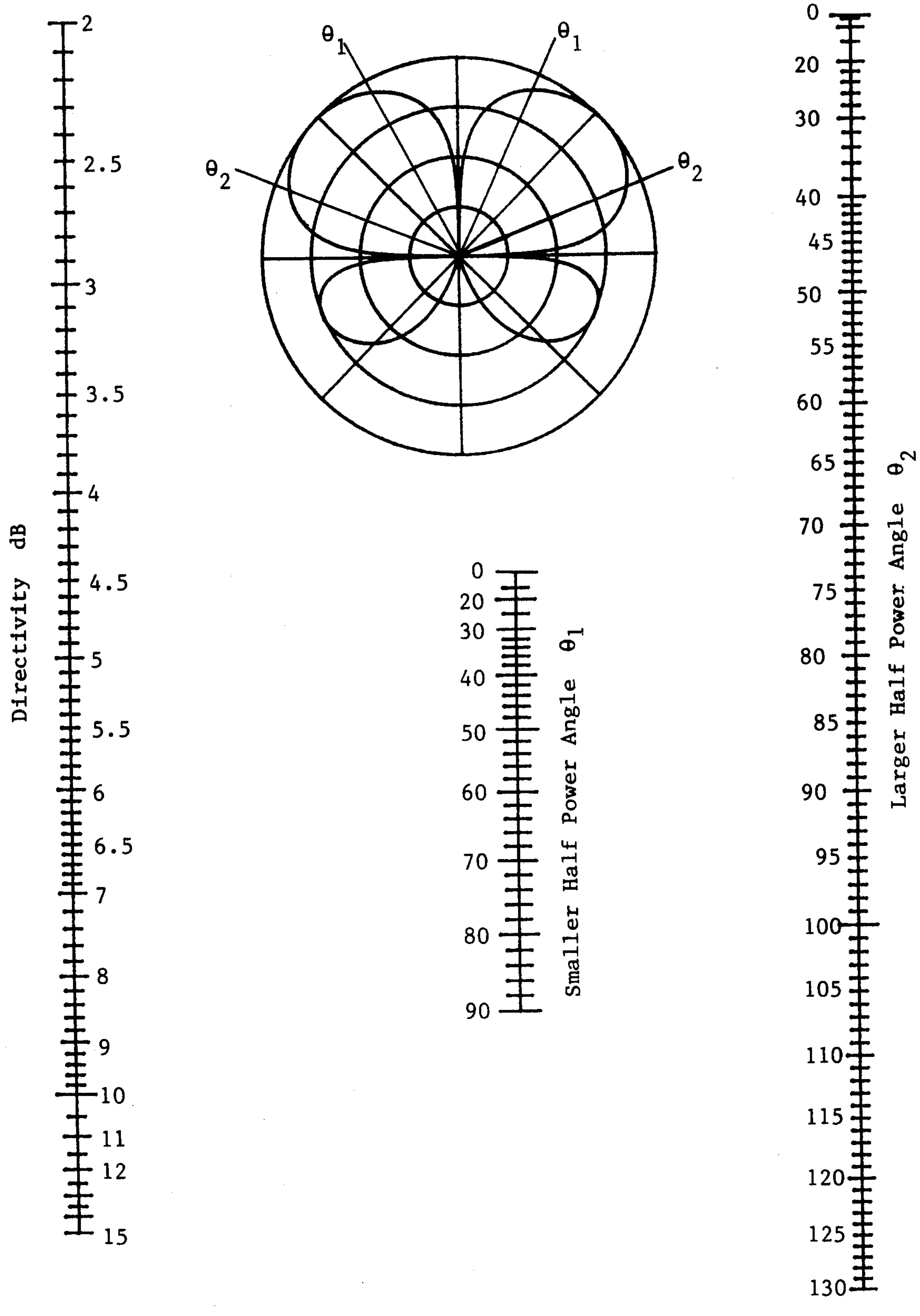
A function similiar to the conical beam approximation can be used to approximate the pattern. The following function can approximate the butterfly pattern.

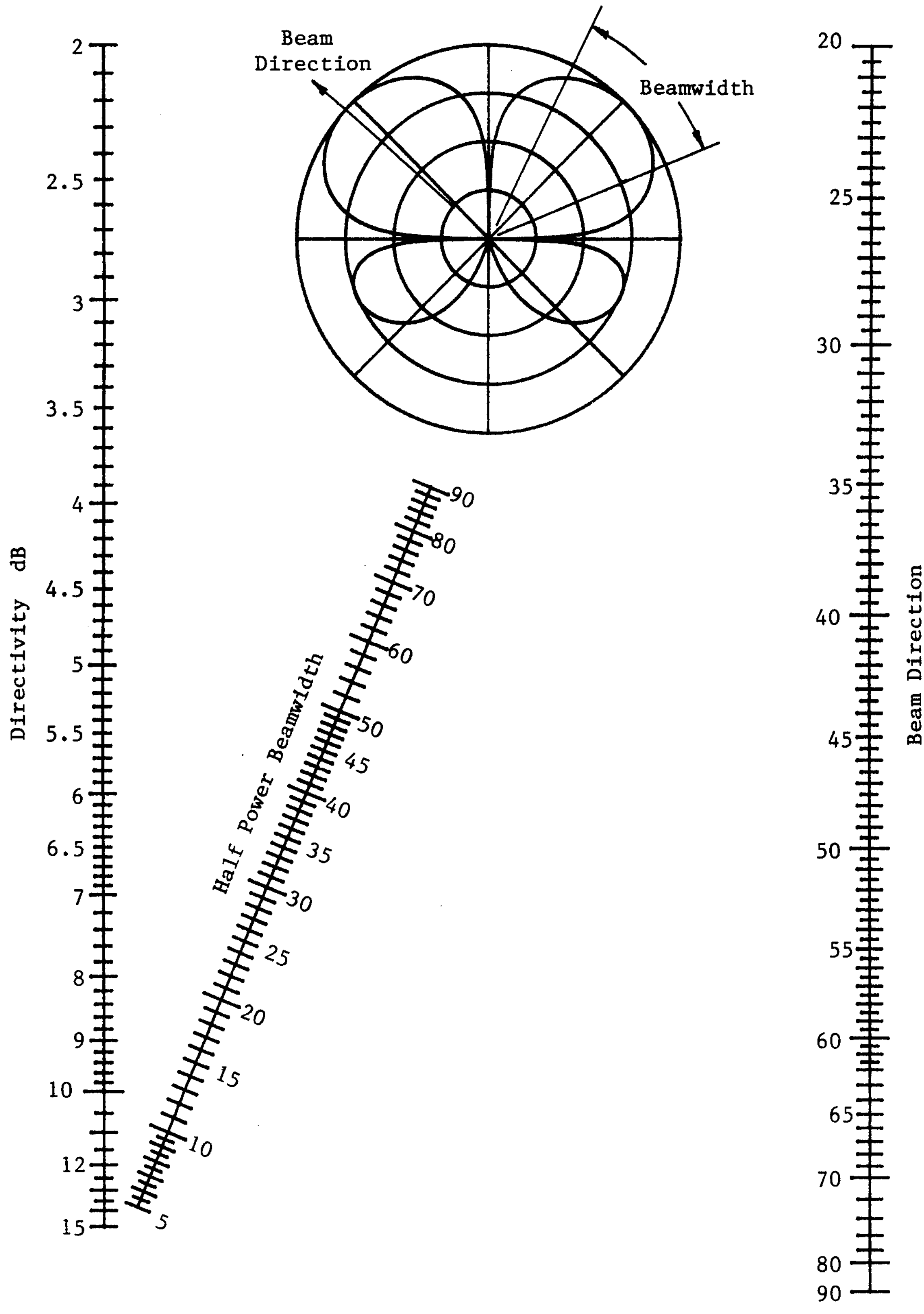
$$U = B\sin^{2M}\left(\frac{\theta}{2}\right)\cos^{2N}\left(\frac{\theta}{2}\right)$$

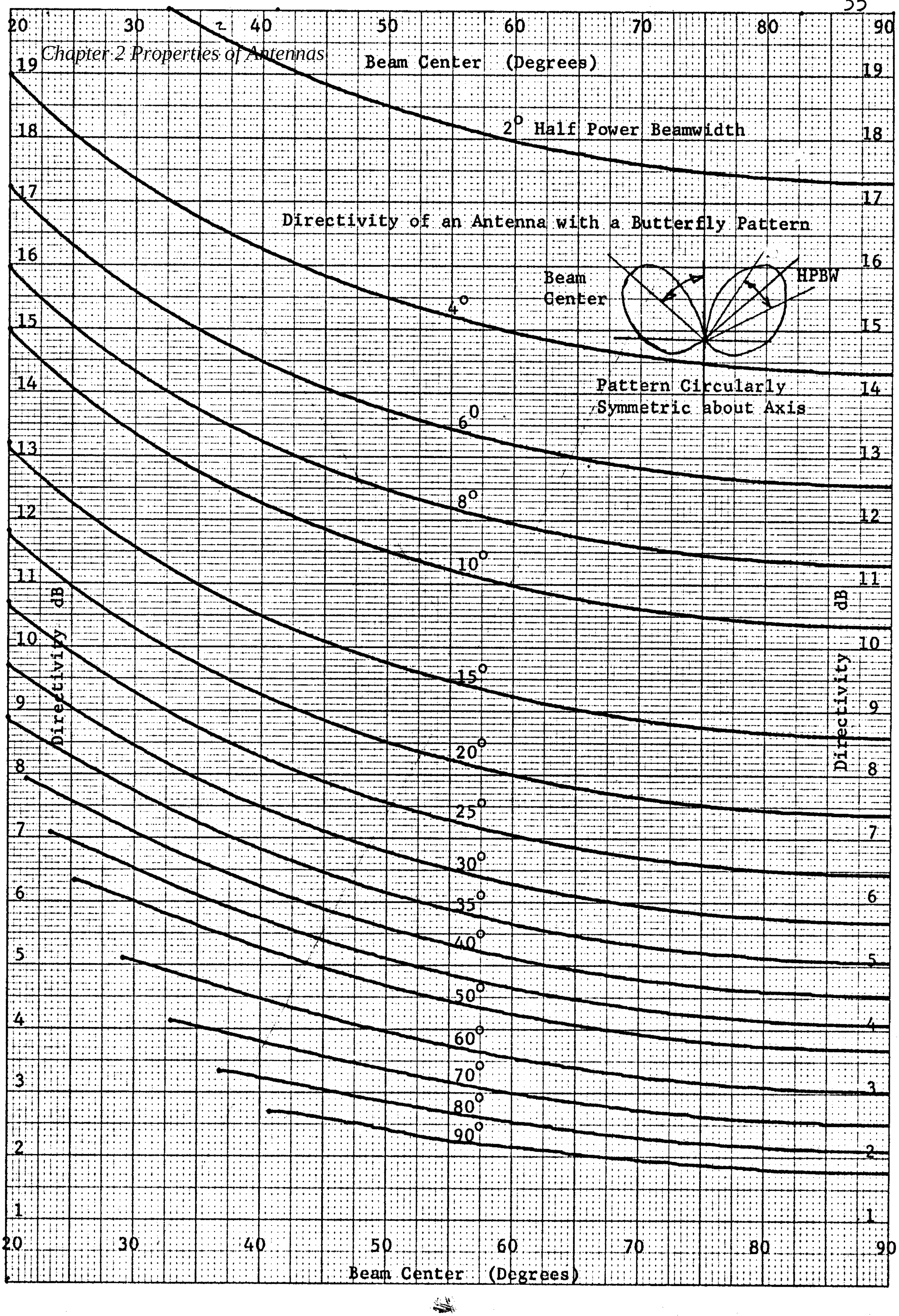
BUTTERFLY PATTERN



Polar Chart No. 127D
SCIENTIFIC-ATLANTA, INC.
ATLANTA, GEORGIA


θ1
θ1
θ2
θ2
Directivity dB
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
8
9
10
11
12
15
Smaller Half Power Angle θ1
0
20
30
40
50
60
70
80
90
Larger Half Power Angle θ2
0
20
30
40
45
50
55
60
65
70
75
80
85
90
95
100
105
110
115
120
125
130


Beam Direction
Beamwidth
Directivity dB
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
8
9
10
12
15
Half Power Beamwidth
90
80
70
60
50
45
40
35
30
25
20
15
10
5
Beam Direction
20
25
30
35
40
45
50
55
60
65
70
80
90


Beam Center (Degrees)
2° Half Power Beamwidth
Directivity of an Antenna with a Butterfly Pattern
Beam Center
HPBW
Pattern Circularly Symmetric about Axis
4°
6°
8°
10°
15°
20°
25°
30°
35°
40°
50°
60°
70°
80°
90°
Directivity dB
Beam Center (Degrees)

Where B is a constant which normalizes the radiation intensity to one. With a little bit of algebra the function can be solved for M and N given the beamwidth and the beam center. This has been done and the radiation intensity has been integrated for a continuum of values. From these the directivity has been calculated and plotted on page 44. The beam center is given on the abscissa, curves are drawn for constant beamwidths, and the directivity in dB is on the ordinate. The pattern on page 42 was drawn using this function.

DIPOLE PATTERNS

A dipole pattern is a special case of the butterfly pattern where the beam center is located at $\theta = 90^\circ$. A formula for the directivity of this pattern was published by N. A. Mcdonald in IEEE Trans. AP March 78. He approximated the radiation intensity by:

$$U(\psi) = \left(\frac{\sin(b\psi)}{b\psi}\right)^2$$

Where ψ is the angle from $\theta = 90^\circ$. The constant b can be related to the beamwidth.

$$b = \frac{159}{\text{HPBW}_{(\text{Deg})}}$$

The average radiation intensity is found by integrating this radiation intensity.

$$\text{Average } \mathbf{U} = \int_0^{\pi/2} \frac{\sin^2 b\psi \cos\psi}{(b\psi)^2}\, d\psi$$

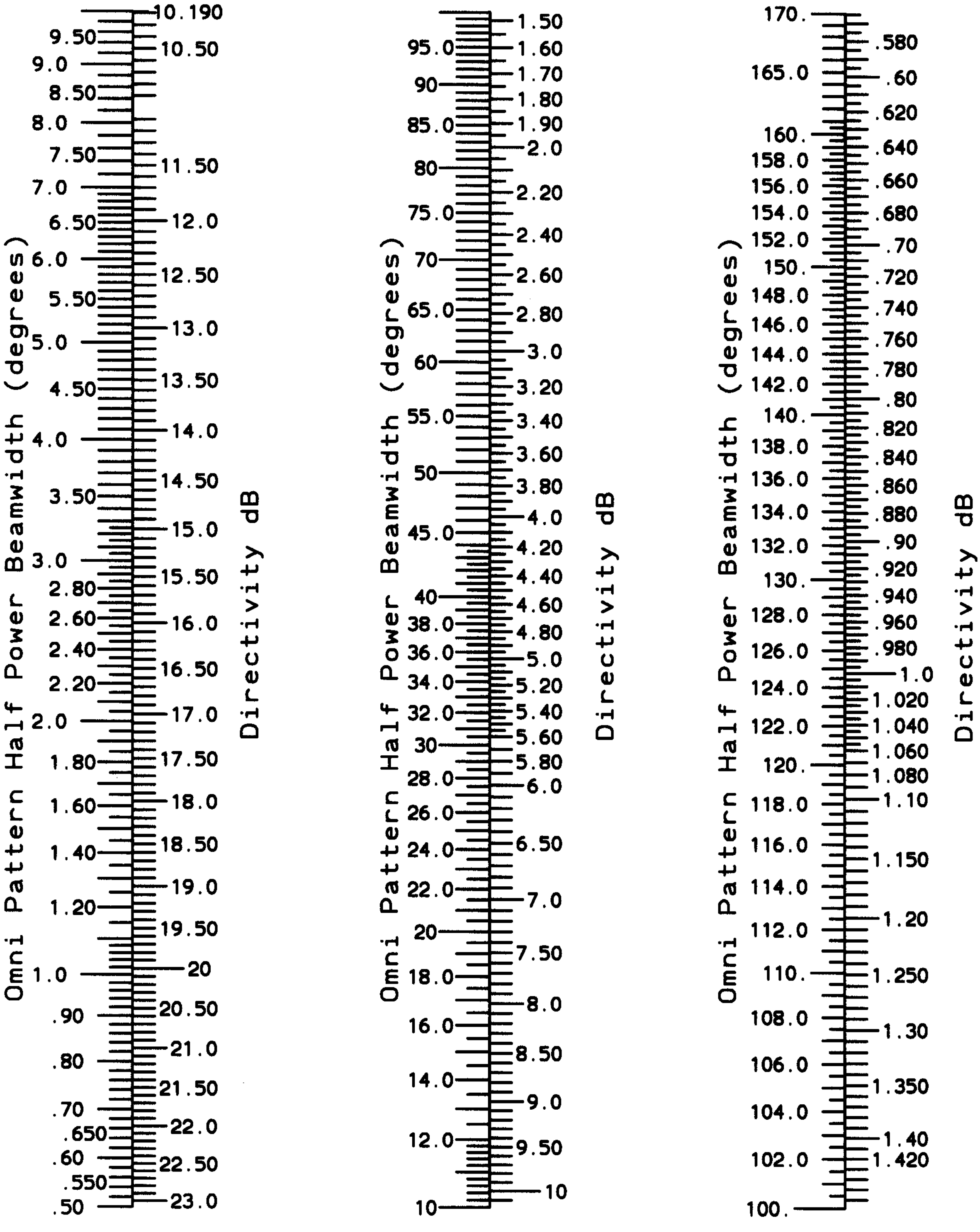
This integral has an exact solution in terms of sine integrals. The directivity is the reciprocal of this integral. The directivity has been calculated using this formula and is plotted on page 45. This pattern is generated by a stack of dipoles on a tower for a broadcast antenna.

UNEQUAL BEAMWIDTHS

Many of these formulas for estimating the directivity from the principle plane pattern cuts assume that the E and H plane beamwidths are equal. When estimating the directivity we could use the mean value of the beamwidth and the beam center for a butterfly pattern, but there is a more reasonable method. Look on the directivity as an estimate of the average radiation intensity.

$$\text{Average Radiation Intensity} = \frac{4\pi\ \text{Umax}}{\text{Directivity}}$$

The problem finding the directivity is calculating or estimating the the integral of the radiation intensity. If we have a pattern with unequal beamwidths, we can take each pattern and estimate the average


Omni Pattern Half Power Beamwidth (degrees)
Directivity dB

radiation intensity by using the curves or nomographs to find the directivity. We have two estimates of the average radiation intensity. The average gives a good estimate and corresponds to using Simpson's rule to numerically integrate over the ϕ variable.

$$\frac{1}{D} = \frac{1}{2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right)$$

Where D_i is the directivity expressed in ratio.

$$D = \frac{2D_1 D_2}{D_1 + D_2}$$

GAIN

The gain of an antenna is another measure of the ability of an antenna to concentrate the input power into radiation in a given direction. The gain is defined as the required power flow into an isotropic radiator to give the same radiation intensity divided by the power flow into the antenna to give the radiation intensity. Gain is understood to be at the maximum radiation intensity. The term directive gain is defined like directive directivity; it is the gain relative to an isotropic radiator in an arbitrary direction. The power into an antenna is the sum of the radiated power and the losses in the antenna including the reflected power at the input connector. The ratio of the power radiated to the input power is called the efficiency of the antenna. The radiated power can be found from the surface integral of the radiation intensity.

$$P_r = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi) \sin\theta \, d\theta \, d\phi$$

The efficiency is given the symbol: η_e . The input power is P_r/η_e
The power flow into an isotropic radiator to give the same radiation intensity is $4\pi U_1 = P_o$

$$\text{Gain} = \frac{P_o}{P_1} = \frac{\eta_e 4\pi U_1}{\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta \, d\theta \, d\phi}$$

This can be recognized as η_e (Directivity)

The gain is a measure of the increased power density of the radiation over the level that would be present if the antenna radiated equally in all directions (an isotropic radiator). When the gain is defined this way, it is said to be the gain relative to an isotropic radiator. The other common gain reference is the half wave dipole. We will see later that the half wave dipole has a gain of 2.15 dB. The half wave dipole is the reference for broadcast antennas and is used in older texts and articles.

EFFECTIVE AREA

The effective area is a measure of the energy captured from a passing wave by an antenna and delivered to the receiver load on the terminals of the antenna.

If an antenna is in a uniform field (plane wave), the effective physical area captures the energy. This energy is divided up as follows: part of the energy is dissipated in the material as I^2R losses, the rest of the energy is delivered to the input terminals of the antenna. If the antenna is mismatched to the transmission line, then part of the energy is reflected at the terminals. We will consider these reflections when we discuss transmission line relations. The energy that passes into the transmission line feeding the antenna is considered delivered to the receiver.

What happens to the energy reflected by the transmission line feed? It travels back through the antenna and part of it gets dissipated in the I^2R material losses. The rest of it is radiated by the antenna. This reradiated energy is distributed in the pattern of the antenna. Since we have accounted for all the energy entering the antenna over the physical area, this reradiated energy is all that is left. The antenna will cast a shadow behind it whose "darkness" depends on the match of the antenna and the level of the radiation pattern of the antenna in that direction.

PATH LOSS

We have considered the radiation of an antenna, which is the gain biased radiation intensity, and we have considered the effective area of an antenna, which is the receiving characteristic. Now it is time to consider the two together. Suppose we have two antennas far away from each other so that we are in the far field. If one of the antennas is radiating we can find the power density at the receiving antenna.

$$S_r = \frac{P_t g_1(\theta, \phi)}{4\pi R^2}$$

The Poynting vector magnitude is equal to the input power of the antenna, P_t, divided by the area of a sphere, $4\pi R^2$ at a distance R (the distance between the antennas), if it radiates isotropically, and times the gain function in the direction between the two antennas. The power received at the terminals of the second antenna will be the effective area of the antenna times the energy density.

$$P_r = A_2 S_r = \frac{P_t A_2 g_1(\theta, \phi)}{4\pi R^2}$$

The ratio of the power received to the power transmitted at the terminals of the antenna is the path loss between two antennas.

$$\frac{P_r}{P_t} = \frac{A_2 g_1(\theta, \phi)}{4\pi R^2}$$

If we consider antenna 2 to be radiating and antenna 1 the receiver, then we will get a similiar formula for the path loss from antenna 2 to antenna 1.

$$\frac{P_r}{P_t} = \frac{A_1\, g_2(\theta, \phi)}{4\pi R^2}$$

When we consider reciprocity, we will see that path loss between the two antennas is the same regardless of which one is the transmitting antenna. Combining the two equations above, the following relationship is found.

$$\frac{g_1}{A_1} = \frac{g_2}{A_2}$$

This is true for any antenna. The ratio of the effective area to the gain equals a constant. If we consider the transmission between two large apertures, the constant can be found. We will not do that now, but the result is:

$$\frac{g}{A} = \frac{4\pi}{\lambda^2}$$

Now we have a relation between gain and effective area.

$$A_{eff} = \frac{g\lambda^2}{4\pi}$$

The transmission between two antennas can be given in terms of the gain of the two antennas.

$$\frac{P_{REC}}{P_{TRANS}} = g_1 g_2 \left(\frac{\lambda}{4\pi R}\right)^2$$

The gain of a large aperture can be found approximately from the area.

$$g = \frac{4\pi A}{\lambda^2}$$

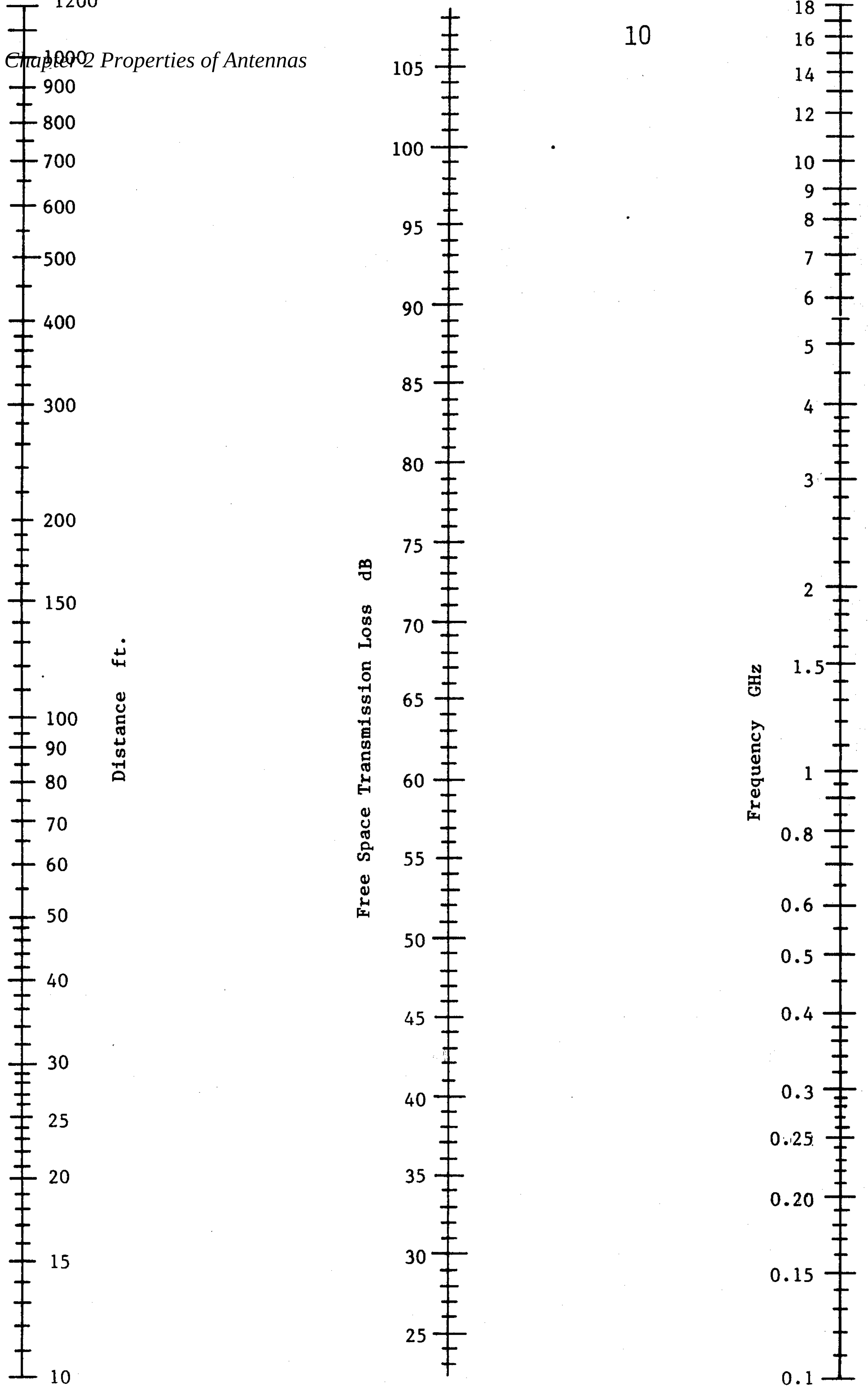
This equation usually over estimates the gain because it assumes uniform amplitude and phase over the antenna aperture. The transmission can be expressed in terms of the effective areas as well.

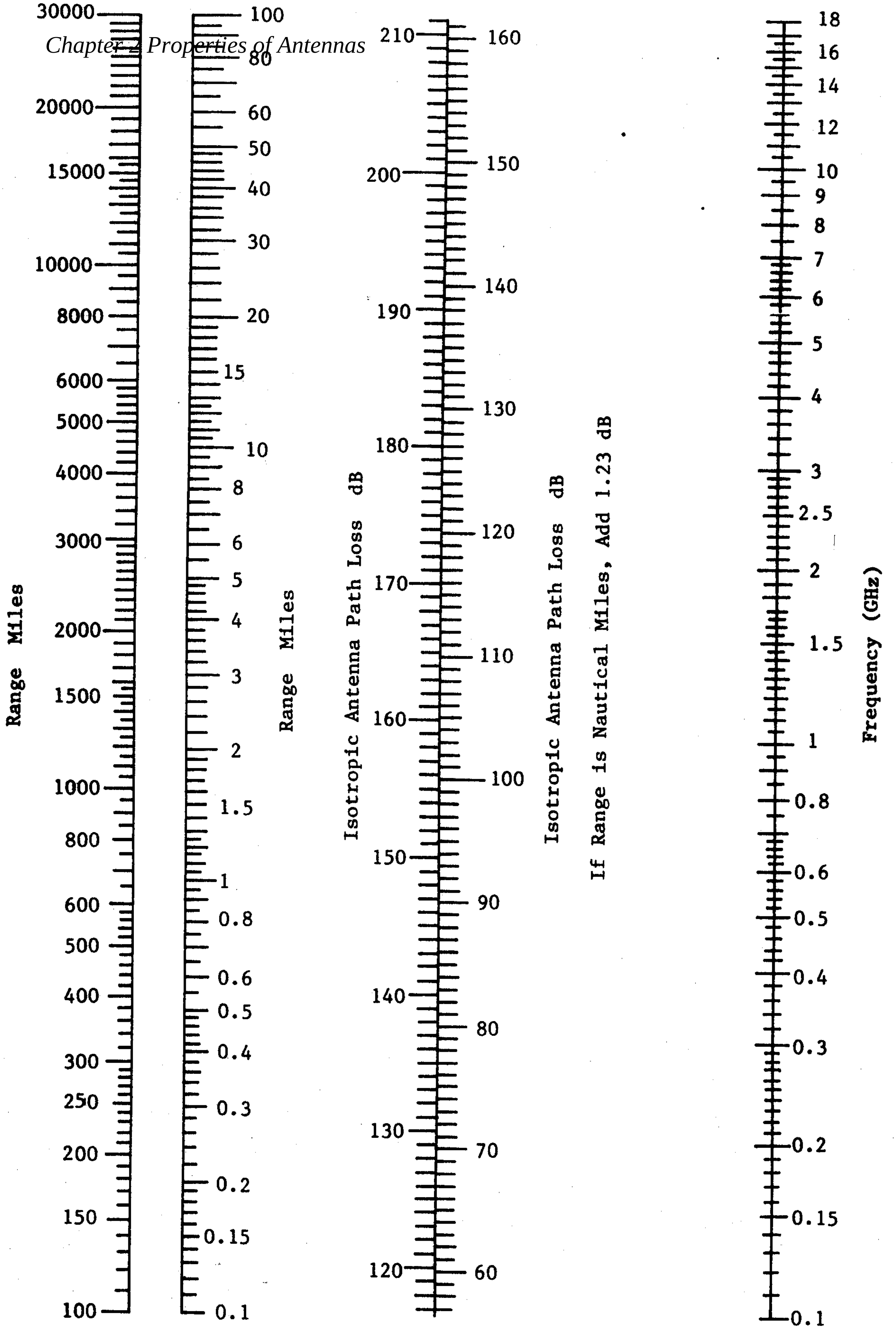
$$\frac{P_{REC}}{P_{TRANS}} = \frac{A_1 A_2}{\lambda^2 R^2}$$

The transmission path loss for two isotropic antennas is given on pages 49 and 50. To find the transmission between two antennas, we first use the nomographs to find the transmission between two isotropic antennas at the frequency and range. The gain of the two antennas expressed in dB then must be substracted from this value to get the path loss.

Instead of using the nomographs to find the path loss between two isotropic antennas, we can use the following formula. The constants have been evaluated.

Path Loss (dB) = 36.58 + 20 Log(R F)


Distance ft.
Free Space Transmission Loss dB
Frequency GHz


Range Miles
30000
20000
15000
10000
8000
6000
5000
4000
3000
2000
1500
1000
800
600
500
400
300
250
200
150
100
Range Miles
100
80
60
50
40
30
20
15
10
8
6
5
4
3
2
1.5
1
0.8
0.6
0.5
0.4
0.3
0.2
0.15
0.1
Isotropic Antenna Path Loss dB
210
200
190
180
170
160
150
140
130
120
Isotropic Antenna Path Loss dB
160
150
140
130
120
110
100
90
80
70
60
If Range is Nautical Miles, Add 1.23 dB
Frequency (GHz)
18
16
14
12
10
9
8
7
6
5
4
3
2.5
2
1.5
1
0.8
0.6
0.5
0.4
0.3
0.2
0.15
0.1

In the path loss formula R is measured in miles and F is the frequency in MHz.

We need to consider the transmission between two apertures as the frequency is changed. If we assume that the effective aperture area remains constant such as in the case of a horn or parabolic dish reflector, then for a constant range the transmission between the two antennas will increase with the square of frequency.

$$\frac{P_{REC}}{P_{TRANS}} = \frac{A_1 A_2}{R^2} \frac{1}{\lambda^2} = \frac{A_1 A_2}{R^2} \left(\frac{F}{C}\right)^2 = B F^2$$

Where B is a constant and C is the velocity of light. We can understand this if we consider the transmission between an isotropic source and an aperture antenna. The isotropic antenna will radiate the same energy density (Poynting vector magnitude) at the aperture antenna regardless of the frequency of operation. The aperture antenna will then capture the same total energy as the frequency is changed. An isotropic source transmitting to an aperture antenna will have a flat frequency response. When transmitting between two apertures, the gain of the source antenna will increase with frequency which we can see from the formula relating the gain to the effective aperture. The gain of an aperture antenna increases as the square of the frequency. Because the source antenna gain increases as the square of frequency, the transmission between two apertures increases as the square of the frequency. Gain is measured relative to an isotropic antenna.

WHY USE AN ANTENNA?

We know that there are many times when we must use an antenna. These are the times when to use anything else is impossible, for example, communications with a missile, or over rugged mountain terrain. In these cases we cannot, or would not, run a cable. But are there times when we would utilize an antenna even over level ground? When we look at the large path losses of antennas we begin to think it is always better to run a cable if possible. Not so.

Example. Suppose we want to establish a link at 3 GHz over land. To get the lowest loss we can use waveguide to carry the message. The waveguide we would choose has only 31.7 dB/mile loss. An alternate link would use two antennas each with a gain of 10 dB. The path loss for this combination is 86 dB for a one mile link. It seems as though there is no comparison. What happens with a 2 mile link? The waveguide run would have twice the loss of the one mile link or 63.4 dB. The two antennas have a loss of 92 dB for the two mile link. The antenna link loses only 6 dB for a doubling of the link distance. For a 3 mile link the waveguide run has a loss of 95.1 dB and the antenna pair link has a loss of 96 dB. The waveguide run is losing fast. For any increase in the distance the antenna system is far better than the very low loss waveguide. In a 6 mile link the waveguide run would have 190.2 dB of loss; the antenna system only 102 dB. For the same loss as the 6 mile waveguide run, the antennas can link about 160 K miles.

Example. The second example of the advantages of antennas comes from a problem at the antenna range. There is a 600 foot outside antenna range. The receiver which is used to measure the patterns of the antennas needs a sample of the signal to phase lock the signal and the local oscillator at a 45 MHz difference. One of the proposed methods was to run a cable through the conduit to the signal source which is 600 ft. down range and couple off some of the signal. The usual cable that is used around the range to carry signals is RG/U 115. The range is to be used at 2 GHz. At 2 GHz the cable has a loss of about 11 dB/ 100 ft; the loss of the run would be 66 dB. If a 10 dB coupler is used at the signal source, then the power loss to the receiver would be 76 dB. Since the source would transmit 100 mw (20 dBm), the signal at the receiver would be -56 dBm which is enough to lock the receiver. None of us wanted to pull 600 ft of cable through the conduit. Instead of pulling the cable we decided to put up a standard gain horn on a small stand out of the way of the measurement but still in the main beam of the source antenna. The source antenna is a 10 ft. parabolic dish antenna which we figure has at least a gain of 31 dB (assuming 30 percent efficiency). From the nomograph of path loss we get an isotropic path loss equal to 84 dB. The standard gain horn has a gain of about 15 dB. The total transmission loss between the source antenna and the reference channel horn is 84 dB - 31 dB - 15 dB = 38 dB. Not only do we not have to pull the cable, we must put a 20 dB pad on the horn so that we do not saturate the receiver (-30 dBm). Even when we have a short run it is sometimes better to transmit the power using antennas instead of cables.

RECIPROCITY

Under certain conditions the transmission between two antennas is the same regardless of which one is the transmitting antenna. Look at two antennas.

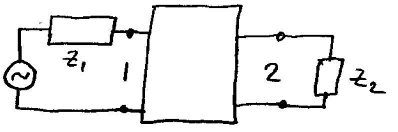
The characteristics of the two antennas above and their interrelationship can be described by an impedance matrix.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Reciprocity states: Any network constructed of linear isotropic materials has a symmetrical impedance (or admittance) matrix.

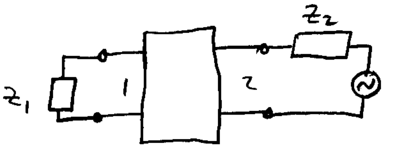
$$Z_{12} = Z_{21}$$

Isotropic means that the characteristics of the material do not depend on direction. A linear material has properties which do not depend on the level of the fields applied on it. Suppose antenna 1 has a source impedance of Z_1 and antenna 2 has a load impedance of Z_2. The response of the two port network of the two antennas and the space between them is:



$$\underset{1 \text{ to } 2}{\text{TRANSMISSION}} = \frac{4\,\mathrm{Re}(Z_1)\,\mathrm{Re}(Z_2)\,|Z_{21}|^2}{|(Z_{11}+Z_1)(Z_{22}+Z_2) - Z_{12}Z_{21}|^2}$$

Now if we interchange the transmitter and the receiver and make the source impedance Z_2 to match antenna 2 and the receiver load Z_1 to match antenna 1, then we can find the transmission from antenna 2 to antenna 1.



$$\underset{2 \text{ to } 1}{\text{TRANSMISSION}} = \frac{4\,\mathrm{Re}(Z_1)\,\mathrm{Re}(Z_2)\,|Z_{12}|^2}{|(Z_{11}+Z_1)(Z_{22}+Z_2) - Z_{12}Z_{21}|^2}$$

Since $Z_{12} = Z_{21}$ by the reciprocity theorem, the power tranfer between the two antennas is identical no matter which one is used for the transmitting antenna.

No mention of the far field has been made in this development. The result is valid in the near field as well as the far field. The second point is when the pattern of an antenna is measured with a probe antenna on a circular path relative to the antenna, the measured pattern is the same whether the antenna under test is transmitting or receiving. When measuring a pattern, the measurement is the transmission between the antennas for various angles. The receive and transmit patterns are the same.