

POLARIZATION

Polarization is the second consideration of the antenna response. Not only must an antenna direct the energy density in certain desired directions, but it must do so with the proper polarization. The transmitting antenna and receiver antenna must be matched in polarization. If they are not, then the signal transmitted will be greatly reduced. We will consider the two most used polarizations, linear and circular, as special cases of the general elliptical polarization.

The polarization of a wave is defined as the direction of the electric field vector. If we have a single electromagnetic wave as was considered on pages 13 and 14, then we would say that the wave is linear polarized and the polarization is aligned with the X axis. We only derived a wave that was polarized with the electric field on the X axis. The equation can also be solved for a wave that is polarized with the electric field on the Y axis. We have assumed that the wave is traveling in the Z direction. The spherical coordinate system could be used just as well. In that case the two polarizations could be aligned with the \bar{a}_θ and \bar{a}_ϕ unit vectors. We will use the rectangular coordinate system and plane waves, but the results hold equally for spherical waves. It will be easier to follow the development.

Remember in the far field that the electric and magnetic fields are orthogonal (perpendicular) and that the magnetic field is related to the electric field by the impedance of free space. All this means that we can ignore the magnetic field and concentrate only on the electric field. The electromagnetic wave travels in a direction orthogonal to the fields which we will take to be the Z axis. In that case the electric field can be expressed:

$$\bar{E} = E_x \bar{a}_x + E_y \bar{a}_y$$

Both E_x and E_y are phasors which we can also express

$$E_x = |E_x| e^{j\phi_x} \quad E_y = |E_y| e^{j\phi_y}$$

We will express the electric field:

$$\bar{E} = E_x (\bar{a}_x + \hat{\rho}_L \bar{a}_y) \quad \hat{\rho}_L = E_y/E_x = \rho_L e^{j\delta_L}$$

$\hat{\rho}_L$ is called the linear polarization ratio.

Let us normalize the equation and consider the equation of the tip of the electric field vector. To do this we must add back the ωt term and take the real part.

$$X = \cos(\omega t - \beta z) \quad Y = \rho_L \cos(\omega t - \beta z + \delta_L)$$

We can expand the equation for Y by using the trigonometry identity for the cosine of the sum of two angles.

$$Y = \rho_L (\cos(\omega t - \beta z) \cos \delta_L - \sin(\omega t - \beta z) \sin \delta_L)$$

We can substitute X for $\cos(\omega t - \beta z)$ in the equation for Y.

$$Y = \rho_L (X \cos \delta_L - \sin(\omega t - \beta z) \sin \delta_L)$$

Now let us solve this equation for the $\sin(\omega t - \beta z)$ term.

$$\sin(\omega t - \beta z) = X \cos \delta_L / \sin \delta_L - Y / (\rho_L \sin \delta_L)$$

We now make use of the trigonometry identity for the sum of the squares of the sine and cosine.

$$\sin^2(A) + \cos^2(A) = 1$$

Using the equation above for the sine term and the fact $X = \cos(\omega t - \beta z)$ we can formulate the following equation.

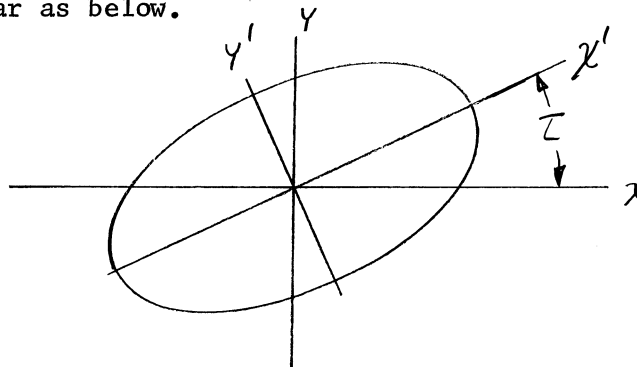
$$(X \cot \delta_L - \frac{Y}{\rho_L \sin \delta_L})^2 + X^2 = 1$$

When we collect the terms in different powers of X and Y, the following equation results.

$$X^2(1 + \cot^2 \delta_L) - \frac{2XY \cos \delta_L}{\rho_L \sin^2 \delta_L} + \frac{Y^2}{\rho_L^2 \sin^2 \delta_L} = 1$$

This equation is in the general form: $A X^2 + B X Y + C Y^2 = 1$ which is the equation of an ellipse.

When looking at an electromagnetic wave from the Z axis of a wave traveling in the Z direction, the tip of the electric field vector will appear to rotate in an ellipse. There are two special cases of the ellipse. The first is when the ellipse collapses into a straight line; this is called linear polarization. In this case the phase angle of the linear polarization ratio will be 0° or 180° . Notice that the linear polarization ratio is like impedance because it is also the ratio of two phasors and is trapped in the frequency domain. The second special case is circular polarization. In this case the ellipse expands until it becomes a circle. For circular polarization the phase of the linear polarization ratio is either $+90^\circ$ or -90° . In general the polarization ellipse will appear as below.



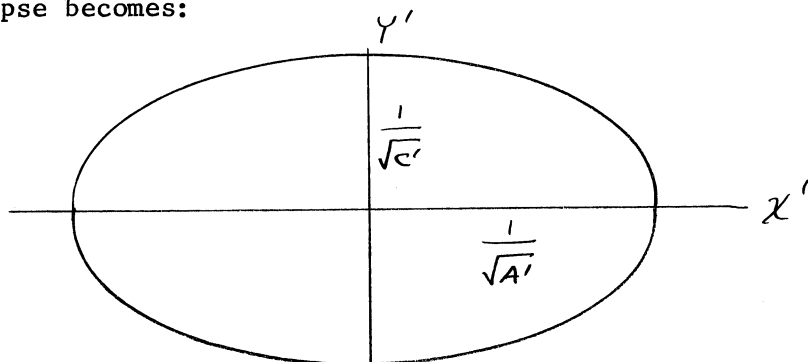
τ is called the tilt angle of the ellipse. If it was assumed that the antenna was to be measured as a linear antenna, the co-polarization direction would be aligned with the X' axis and the tilt angle would be the angle of the linear polarization. Since the polarization is not strictly linear but elliptical, the minimum linear response of the antenna is called the cross polarization response. The cross polarization direction is aligned with the Y' axis. If we rotate the coordinate system by the angle τ , we will eliminate the XY term.

$$X = X' \cos \tau - Y' \sin \tau \quad Y = X' \sin \tau + Y' \cos \tau$$

If we substitute these into the equation for the ellipse, we can find the tilt angle. We could do this which is the traditional way of handling this and a great exercise in algebra and trigonometry, but it is easier to come back to it with circular polarized waves. In the end we will have an equation in the rotated coordinates.

$$A' X'^2 + C' Y'^2 = 1$$

With τ , the tilt angle determined by the requirement that the B' term = 0 and $A' \leq C'$. In the rotated coordinate system the polarization ellipse becomes:



For this diagram we define the axial ratio to be the maximum linear response divided by the minimum linear response.

$$\text{Axial Ratio} = \frac{1/\sqrt{A'}}{1/\sqrt{C'}} = \sqrt{\frac{C'}{A'}} = \frac{MAX E}{MIN E}$$

The axial ratio is the ratio of two electric fields. The power in a wave is proportional to the square of the electric field magnitude. The axial ratio can be given in decibels.

$$\text{Axial Ratio (dB)} = 20 \log \sqrt{\frac{C'}{A'}} = 10 \log \frac{C'}{A'}$$

The axial ratio is usually associated with circularly polarized waves. It may also be associated with linear polarization but cross polarization is the term used with linear antennas. The axial ratio is the reciprocal of the cross polarization response.

CIRCULAR POLARIZATION

When the phase of the linear polarization ratio, δ_L , equals $\pm \frac{\pi}{2}$ (90°) and the magnitude of the linear polarization ratio is one, then the equation for the polarization ellipse becomes.

$$X^2 + Y^2 = 1$$

This is the equation of a circle, the special case called circular polarization. Let us go back and consider the parametric equations of the polarization ellipse which is now a circle.

$$Y = \cos(\omega t - \beta z \pm \frac{\pi}{2})$$

$$Y = \mp \sin(\omega t - \beta z)$$

$$X = \cos(\omega t - \beta z)$$

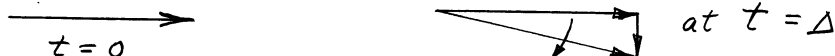
If we position ourselves at $Z = 0$ and at time equals zero, the electric field vector will be aligned with the X axis. We need to look at the two cases separately.

Case 1. $\delta_L = \frac{\pi}{2}$ In this case the equation for Y is given as;

$$Y = -\sin(\omega t - \beta z)$$

The equation for X is: $X = \cos(\omega t - \beta z)$

If we start at zero time and increase time by a small amount, the cosine term, X , will change very little. The sine term will start from zero and become negative. We can see this in the vector diagram below.



The tip of the vector has started to rotate clockwise as we look at the wave coming toward us. Remember that the wave is propagating in the Z axis direction. This circularly polarized wave is called left hand circular polarization because if we curl the fingers of our left hand in the direction of the rotation of the electric field our thumb points in the direction of propagation.

Case 2. $\delta = -\frac{\pi}{2}$ In this case the equation for Y is given as:

$$Y = \sin(\omega t - \beta z)$$

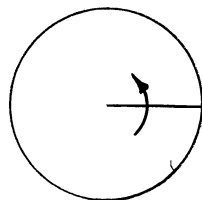
The equation for X is still: $X = \cos(\omega t - \beta z)$

If we start at time zero and increase time by a small amount, the cosine term, X , will again change very little. Y , the sine term, will start from zero and become positive. We draw the vector diagram below.

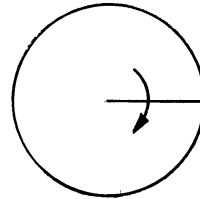


As you may have guessed, case 2 is right hand circular polarization. For this case you use the right hand with the direction of propagation in the direction of the thumb.

The two cases look like this when sitting on the Z axis and watching the waves propagating in the Z direction.



RHC



LHC

What is the rate of rotation of the electric field vector in a circularly polarized wave? If we look at the parametric equations above, we will see that the rotational rate is ωt . That is, the vector rotates once for every period ($1/F$) of the wave. At any instant of time the wave will look like a corkscrew in space, and as the wave propagates, it looks like the screw is rotating.

CIRCULAR POLARIZATION COMPONENTS

We can define a basis set of vectors for polarization using the two circular polarizations. The normalized right and left hand circular component unit vectors are defined in terms of the X and Y basis unit vectors as:

$$\bar{a}_R = \frac{1}{\sqrt{2}} (\bar{a}_x - j\bar{a}_y) \quad \bar{a}_L = \frac{1}{\sqrt{2}} (\bar{a}_x + j\bar{a}_y)$$

We have used the equation for the linear polarization ratio as such:

$$\rho_L e^{j\delta_L} = e^{j(\pm\frac{\pi}{2})} = \pm j$$

The term $1/\sqrt{2}$ comes from the normalization of the magnitude of the vectors to one. We will show that this is a valid basis set for the two dimensional space of polarizations, but first we have to look at the scalar or dot product of two vectors with complex coefficients.

DOT OR SCALAR PRODUCT OF TWO VECTORS WITH COMPLEX COEFFICIENTS

If we are given two vectors with complex coefficients, such as phasor vectors, then the scalar product is defined.

$$\bar{A}^* \cdot \bar{B}$$

Where \bar{A}^* is the complex conjugate of \bar{A} .

Using the above definition for the vector dot product for vectors with complex coefficients, we can take the dot product of the unit vector of right circular polarization.

$$\begin{aligned}\bar{a}_R^* \cdot \bar{a}_R &= \frac{1}{\sqrt{2}} (\bar{a}_x + j\bar{a}_y) \cdot \frac{1}{\sqrt{2}} (\bar{a}_x - j\bar{a}_y) \\ &= \frac{1}{2} (1 - (j)(j)) = 1\end{aligned}$$

Similarly the dot product of the unit vectors of left hand circular polarization is:

$$\bar{a}_L^* \cdot \bar{a}_L = 1$$

If we take the dot product of the two vectors \bar{a}_L and \bar{a}_R , then we will get the projection of the right hand circular polarization on to the left hand circular polarization.

$$\begin{aligned}\bar{a}_L^* \cdot \bar{a}_R &= \frac{1}{\sqrt{2}} (\bar{a}_x - j\bar{a}_y) \cdot \frac{1}{\sqrt{2}} (\bar{a}_x - j\bar{a}_y) \\ &= \frac{1}{2} (1 + (j)(j)) = 0\end{aligned}$$

Because the dot product of the two vectors is zero we say that the vector \bar{a}_L is orthogonal to \bar{a}_R and the pair form an orthogonal basis set for antenna polarization. Using this basis we can express the electric field:

$$\bar{E} = E_R \bar{a}_R + E_L \bar{a}_L$$

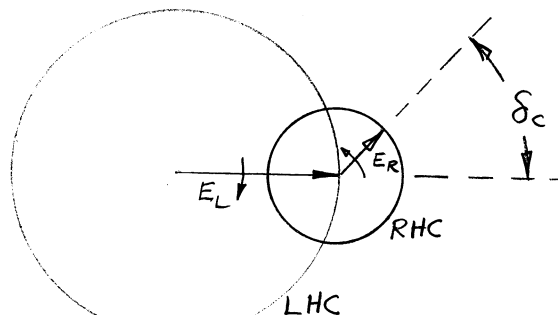
Where E_R and E_L are phasors which can be expressed $E_R = |E_R| e^{j\phi_R}$, etc., and \bar{a}_R and \bar{a}_L are unit vectors. We can also express the electric field by:

$$\bar{E} = E_L (\bar{a}_L + \hat{\rho}_c \bar{a}_R)$$

$$\hat{\rho}_c = E_R / E_L \quad \hat{\rho}_c = \rho_c e^{j\delta_c}$$

$\hat{\rho}_c$ is called the circular polarization ratio.

Let us look at a predominately left hand circular polarized wave when the time and space give a phase of zero. We can draw the polarization as two circles as below. Each circle rotates with a rate of ωt in opposite directions with the center of the right hand circular polarization circle moving on the end of the vector of the left hand polarization.



δ_c is the phase angle of the circular polarization ratio and is also the phase difference between the right and left hand circular components. The maximum signal occurs when the right and left hand circular components are aligned. If we look at the diagram and consider time moving backwards, then as time moves backwards the left hand component will rotate CCW and the right component CW. The angle between the two vectors will decrease at twice the rotation rate of either one of them. When the vectors are lined up, the angle of the left hand component will be $-\tau$. τ is the tilt angle. Therefore the relation between the tilt angle and the phase of the circular polarization ratio is:

$$\tau = \delta_c / 2$$

RELATIONSHIP BETWEEN X-Y BASES AND CIRCULAR BASES

We will find the relationship between the two sets of bases vectors in two ways. In the first way we equate the two representations and solve the simultaneous equations.

$$E_i (\bar{a}_x + \hat{\rho}_L \bar{a}_y) = \frac{E_R}{\sqrt{2}} (\bar{a}_x - j \bar{a}_y) + \frac{E_L}{\sqrt{2}} (\bar{a}_x + j \bar{a}_y)$$

If we equate the \bar{a}_x and the \bar{a}_y components, we have two equations.

$$E_i = \frac{1}{\sqrt{2}} (E_R + E_L) \quad \hat{\rho}_L E_i = \frac{-j}{\sqrt{2}} (E_R - E_L)$$

From these we can solve for the right and left hand components.

$$\sqrt{2} E_i = E_R + E_L \quad j \sqrt{2} \hat{\rho}_L E_i = E_R - E_L$$

Solve for E_R by adding the two equations and solve for E_L by subtracting the two equations.

$$2 E_R = \sqrt{2} E_i (1 + j \hat{\rho}_L) \quad 2 E_L = \sqrt{2} E_i (1 - j \hat{\rho}_L)$$

$$E_R = \frac{E_i}{\sqrt{2}} (1 + j \hat{\rho}_L) \quad E_L = \frac{E_i}{\sqrt{2}} (1 - j \hat{\rho}_L)$$

The circular polarization ratio is the ratio of the E_R component to the E_L component.

$$\hat{\rho}_c = \frac{E_R}{E_L} = \frac{(1 + j \hat{\rho}_L)}{(1 - j \hat{\rho}_L)}$$

The second method of finding the relationship between the X-Y bases and the circular bases is to project the electric field on to the \bar{a}_R and \bar{a}_L vectors. This is done by using the scalar or dot product.

$$\begin{aligned} E_R &= \bar{a}_R^* \cdot (E_1 (\bar{a}_x + j \hat{\rho}_L \bar{a}_y)) \\ &= \frac{1}{\sqrt{2}} (\bar{a}_x + j \bar{a}_y) \cdot (E_1 (\bar{a}_x + j \hat{\rho}_L \bar{a}_y)) \\ E_R &= \frac{E_1}{\sqrt{2}} (1 + j \hat{\rho}_L) \end{aligned}$$

Similarly for the E_L component.

$$\begin{aligned} E_L &= \bar{a}_L^* \cdot E = \frac{E_1}{\sqrt{2}} (\bar{a}_x - j \bar{a}_y) \cdot (\bar{a}_x + j \hat{\rho}_L \bar{a}_y) \\ E_L &= \frac{E_1}{\sqrt{2}} (1 - j \hat{\rho}_L) \end{aligned}$$

AXIAL RATIO

The term axial ratio is usually associated with circular polarization. Consider the axial ratio using the circular polarization vectors as the bases for describing the electric field. The maximum electric field will occur when the two rotating vectors are aligned along a radius vector.

$$E_{\text{Max}} = E_R + E_L$$

The minimum will occur when the two vectors are opposite along the radius vector. If we assume that the wave is predominately right hand circular polarization, then the minimum is:

$$E_{\text{Min}} = E_R - E_L$$

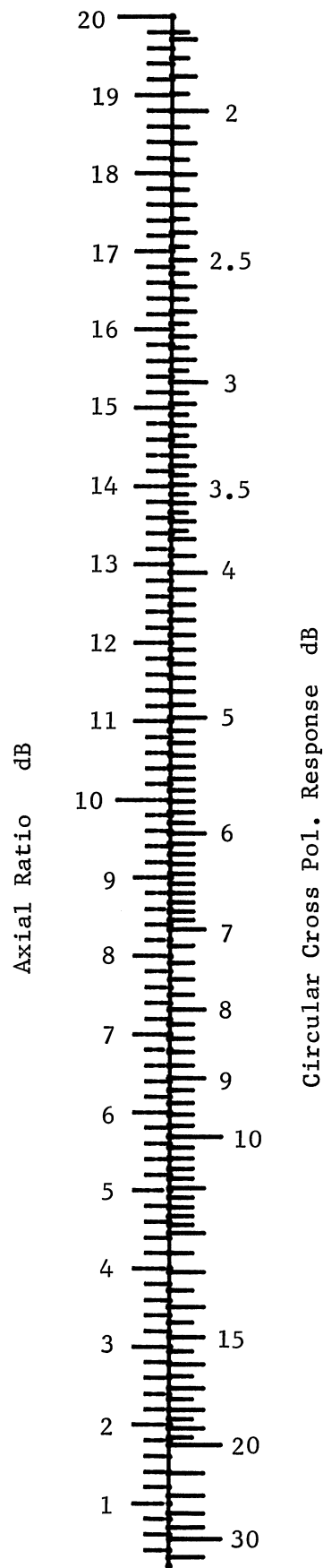
The axial ratio is the ratio of the maximum signal to the minimum signal

$$\text{Axial Ratio} = \frac{E_{\text{MAX}}}{E_{\text{MIN}}} = \frac{|E_R| + |E_L|}{|E_R| - |E_L|} = \frac{|\rho_c| + 1}{|\rho_c| - 1}$$

Note this is the ratio of field magnitudes (voltage). We can express the axial ratio in decibels.

$$\text{AXIAL RATIO (dB)} = 20 \log \left(\frac{E_R + E_L}{E_R - E_L} \right)$$

If an antenna is RHC, the LHC is called the cross polarization of the antenna. The axial ratio is used as a measure of the cross polarization of a circular antenna. An alignment scale on page 62 gives the relationship between axial ratio and the circular cross polarization response.



9/8/80 TAM

ANTENNA RESPONSE

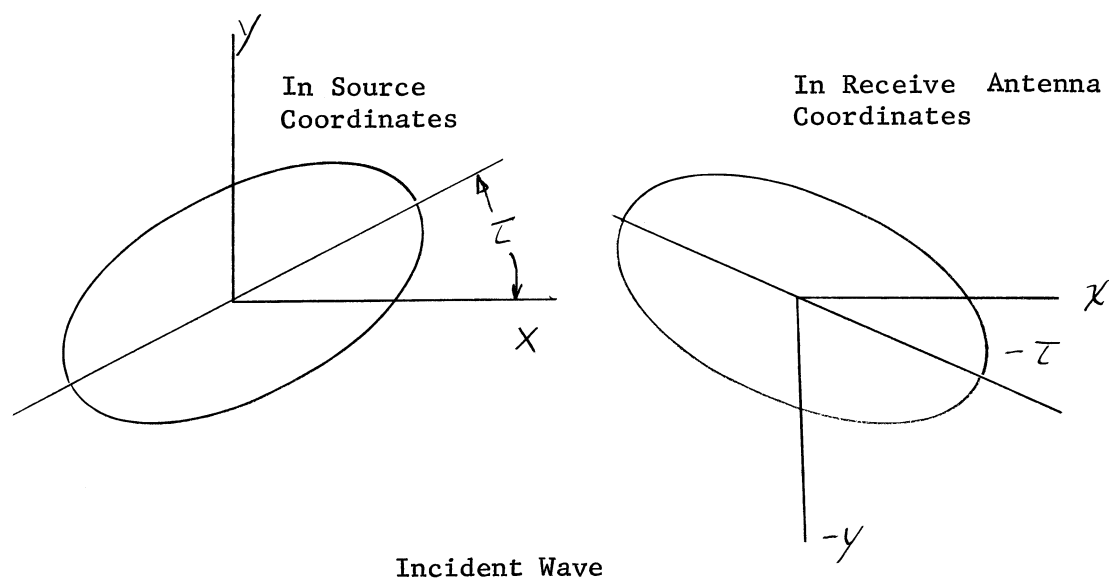
Now that we have a foundation for the polarization of waves, we can discuss the response of an antenna when the polarization of the wave does not match the polarization of the antenna. If an antenna transmits in the Z direction, the wave transmitted by the antenna is:

$$E_A = E_1 (\bar{a}_x + \hat{\rho}_L \bar{a}_y)$$

The incident wave on an antenna is given by

$$E_i = E_2 (\bar{a}_x + \hat{\rho}_{L_2} \bar{a}_y)$$

This wave is in the coordinates of the source antenna. Since the source coordinate system is rotated with respect to the receive antenna, it will be necessary to rotate one of the antenna coordinate systems. The receiving antenna coordinates are held constant and the description of the incident wave is rotated about the X axis.



When the incident wave is expressed in the antenna coordinates, the sign of the tilt angle changes. This implies that the sign of the linear polarization ratio also changes. Changing the sign of a phasor quantity is the same as taking the complex conjugate. Therefore to express an incident wave in terms of the antenna coordinates we must take the complex conjugate of the polarization expressed in the source coordinates.

$$E_i^* = E_2 (\bar{a}_x + \rho_{L_2}^* \bar{a}_y) \quad \text{in antenna coordinates}$$

The response of an antenna to an incident wave is the vector dot product of the incident wave polarization with the polarization

of the antenna. The voltage response of the antenna to the incident wave is:

$$V = V_0 (1 + \hat{p}_L, \hat{p}_L^*)$$

If we normalize the antenna response and the incident wave, then the polarization loss can be found.

$$E_i^* = \frac{a_x + \hat{p}_L^* a_y}{\sqrt{1 + \hat{p}_L^* \hat{p}_L^*}} \quad E_A = \frac{a_x + \hat{p}_L \bar{a}_y}{\sqrt{1 + \hat{p}_L \hat{p}_L^*}}$$

The normalized antenna response becomes

$$\bar{E}_i^*, \bar{E}_A = \frac{1 + \hat{p}_L, \hat{p}_L^*}{\sqrt{1 + \hat{p}_L, \hat{p}_L^*} \sqrt{1 + \hat{p}_L^* \hat{p}_L^*}}$$

The magnitude of the response is the square root of the product of the response times the complex conjugate of the response.

$$\frac{\sqrt{(1 + \hat{p}_L, \hat{p}_L^*)(1 + \hat{p}_L^* \hat{p}_L^*)}}{\sqrt{1 + \hat{p}_L, \hat{p}_L^*} \sqrt{1 + \hat{p}_L^* \hat{p}_L^*}}$$

To find the power response we square the voltage response.

$$\frac{1 + (\hat{p}_L, \hat{p}_L^* + \hat{p}_L^* \hat{p}_L^*) + |\hat{p}_L|^2 |\hat{p}_L^*|^2}{(1 + |\hat{p}_L|^2)(1 + |\hat{p}_L^*|^2)}$$

We need to consider the middle terms of the power response.

$$\hat{p}_L, \hat{p}_L^* = |\hat{p}_L| |\hat{p}_L^*| e^{j(\delta_1 - \delta_2)}$$

$$\hat{p}_L^* \hat{p}_L = |\hat{p}_L| |\hat{p}_L| e^{-j(\delta_1 - \delta_2)}$$

$$\hat{p}_L, \hat{p}_L^* + \hat{p}_L^* \hat{p}_L = |\hat{p}_L| |\hat{p}_L| (e^{j(\delta_1 - \delta_2)} + e^{-j(\delta_1 - \delta_2)})$$

The last expression can be reduced by using the following identity.

$$\cos A = \frac{1}{2} (e^{jA} + e^{-jA})$$

Using this to reduce the equation, the power response can be written.

$$\eta = \frac{1 + |\hat{p}_L|^2 |\hat{p}_L^*|^2 + 2 |\hat{p}_L| |\hat{p}_L^*| \cos(\delta_1 - \delta_2)}{(1 + |\hat{p}_L|^2)(1 + |\hat{p}_L^*|^2)}$$

This is sometimes called the polarization coefficient. It accounts for the loss due to a mismatch of polarization between the incident wave and the antenna.

GENERAL ORTHOGONAL POLARIZATION BASES

Using the polarization coefficient we can define orthogonal polarizations which have no transmission between them or $\Gamma = 0$.

Two polarizations are orthogonal if

$$1) |\hat{\rho}_1| = \frac{1}{|\hat{\rho}_2|}$$

and $2) \delta_1 - \delta_2 = \pm 180^\circ$

It is also true vectorially; if \bar{a}_m and \bar{a}_n are orthogonal then:

$$\bar{a}_m^* \cdot \bar{a}_n = 0$$

Let us consider the response when the source and antenna transmitted waves are expressed in terms of an arbitrary set of orthogonal basis vectors for polarization (normalized so that $\bar{a}_m^* \cdot \bar{a}_m = 1$).

The Antenna $\bar{E}_A = \frac{(\bar{a}_m + \hat{\rho}_A \bar{a}_N)}{\sqrt{1 + \hat{\rho}_A \hat{\rho}_A^*}}$

Incident Wave $\bar{E}_i = \frac{(\bar{a}_m + \hat{\rho}_w \bar{a}_N)}{\sqrt{1 + \hat{\rho}_w \hat{\rho}_w^*}}$

The response is: $\bar{E}_i^* \cdot \bar{E}_A = \frac{(\bar{a}_m^* + \hat{\rho}_w^* \bar{a}_N^*) \cdot (\bar{a}_m + \hat{\rho}_A \bar{a}_N)}{\sqrt{1 + \hat{\rho}_A \hat{\rho}_A^*} \sqrt{1 + \hat{\rho}_w \hat{\rho}_w^*}}$

$$\bar{E}_i^* \cdot \bar{E}_A = \frac{\bar{a}_m^* \cdot \bar{a}_m + \hat{\rho}_w^* \bar{a}_N^* \cdot \bar{a}_m + \hat{\rho}_A \bar{a}_m^* \cdot \bar{a}_N + \hat{\rho}_A \hat{\rho}_w^* \bar{a}_N^* \cdot \bar{a}_N}{\sqrt{1 + \hat{\rho}_A \hat{\rho}_A^*} \sqrt{1 + \hat{\rho}_w \hat{\rho}_w^*}}$$

From the orthogonality and the normalization we have:

$$\begin{aligned} \bar{a}_m^* \cdot \bar{a}_N &= 0 & \bar{a}_N^* \cdot \bar{a}_m &= 0 \\ \bar{a}_m^* \cdot \bar{a}_m &= 1 & \bar{a}_N^* \cdot \bar{a}_N &= 1 \end{aligned}$$

The response reduces to the following using these results:

$$\bar{E}_i^* \cdot \bar{E}_A = \frac{1 + \hat{\rho}_A \hat{\rho}_w^*}{\sqrt{1 + \hat{\rho}_A \hat{\rho}_A^*} \sqrt{1 + \hat{\rho}_w \hat{\rho}_w^*}}$$

This has the same form as the response using X-Y basis vectors. The same analysis which was used with that basis can be repeated for this arbitrary orthonormal basis vectors. The polarization coefficient

becomes:

$$T = \frac{1 + 2|\hat{p}_A||\hat{p}_W| \cos(\delta_A^{MN} - \delta_W^{MN}) + |\hat{p}_A|^2 |\hat{p}_W|^2}{(1 + \hat{p}_A \hat{p}_A^*)(1 + \hat{p}_W \hat{p}_W^*)}$$

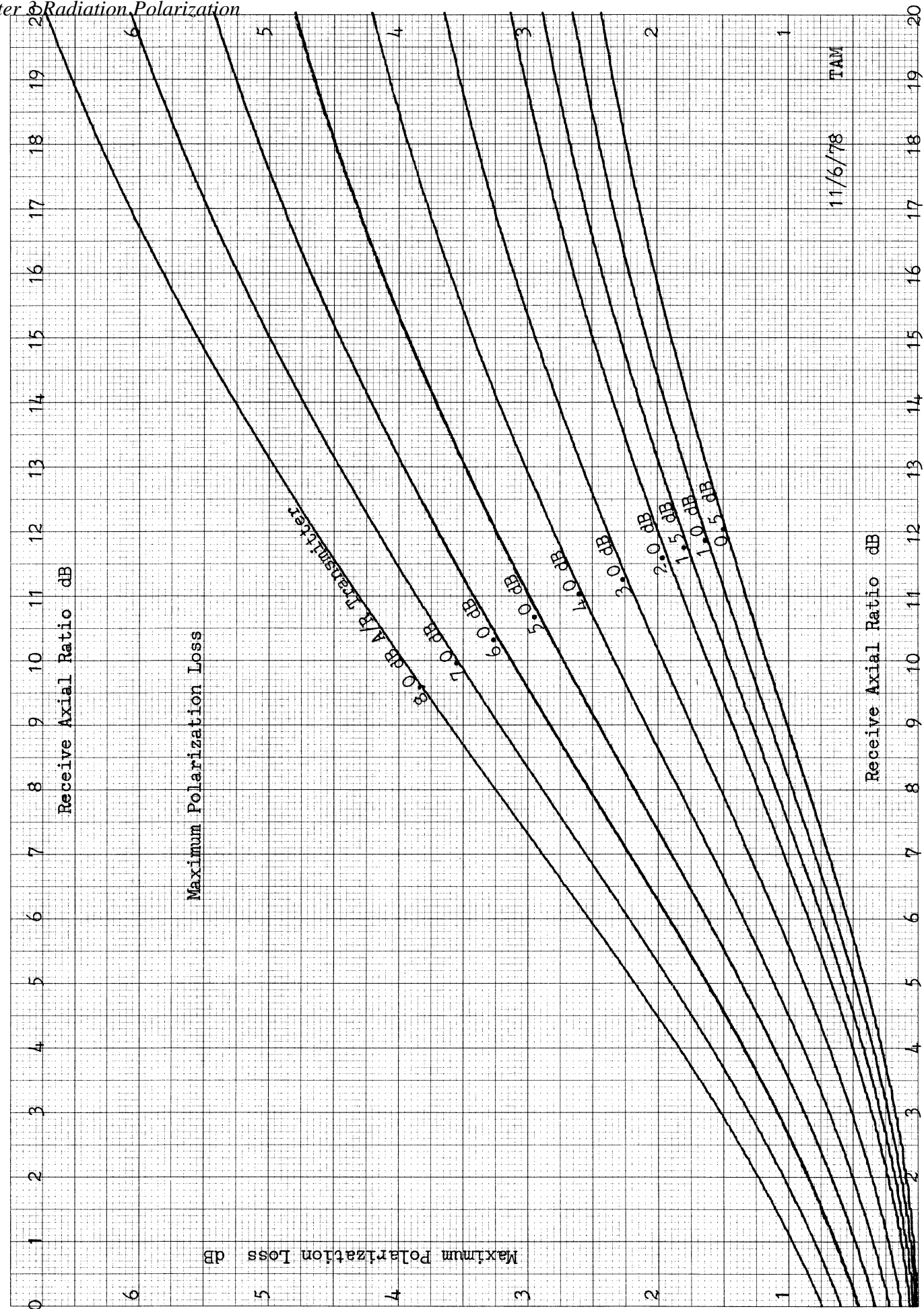
This equation is the exact same form as the rectangular bases equation. For any orthonormal representation we get the same polarization coefficient form for the general basis vectors.

If we use the circular polarization basis vectors, the polarization coefficient has the same form as the polarization coefficient for rectangular basis vectors except we substitute the circular polarization ratio for the rectangular polarization ratio. When we use the circular polarizations for the basis the ratio of the two rotating circles remains the same and only the phase angle changes as one of the antennas is rotated. The magnitude of the circular polarization ratio, \hat{p}_c , remains constant under rotations of the antenna. The magnitude and phase of the linear polarization ratio changes as the antenna is rotated. The circular polarization bases is handy when one of the antennas is rotated arbitrarily.

Circularly polarized antennas are used when we cannot hold constant the angle between the transmit and receive antenna, such as on a satellite spinning. If we rotate one of the circular polarized antennas about the axis between the two antennas, the polarization loss remains constant, although the transmitted phase varies. Since neither the transmitting antenna or the receiving antenna has a perfect axial ratio, we need to know the maximum polarization loss possible between the two antennas. If we take the circular polarization coefficient and use 180° in the cosine term, we get the maximum polarization loss. When we do this we get the curve on page 67 which is the maximum polarization loss given the transmit and receive antenna axial ratios.

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