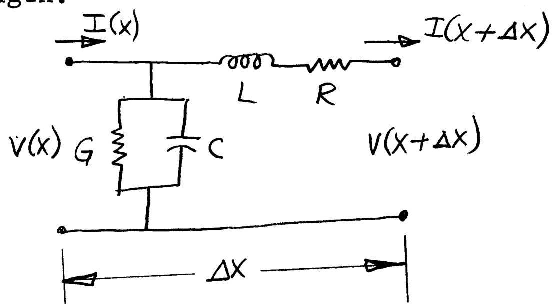
TRANSMISSION LINES

We will cover transmission lines in the traditional development. Much of antenna theory is given in the language of transmission lines and transmission line relations. The two wire transmission line will be covered here and waveguides when we discuss horns. Only the TEM mode will be considered.

A transmission line can be represented by a network which includes the series resistance per unit length, the series inductance per unit length, the shunt conductance per unit length, and the shunt capacitance per unit length.



Above is a circuit diagram for a differential length of transmission line. We can find the differential voltage drop along the differential length

$$\Delta V = V(x+\Delta x) - V(x) = -IZ\Delta x$$

where $Z = R + j\omega L$, and R and L are the series resistance and inductance per unit length. The increment of shunt current associated with a length Δx is

$$\Delta I = -VY\Delta x$$

The line current decreases with length because the current is shunted off by the unit admittances. The admittance per unit length, Y, is equal to the sum of the shunt conductance and capacitance per unit length. As $\Delta x \to 0$ in the limit the equations become:

$$\frac{dV}{dx} = -IZ \qquad \frac{dI}{dx} = -YV$$

Take the derivative of the first equation and substitute the second equation for the derivative of the current and we get an equation only in voltage.

$$\frac{d^2V}{dx^2} = ZYV = \gamma^2 V$$

Similiarly we can do the same with the equations in the reverse order and get an equation in current.

$$\frac{d^2 I}{dx^2} = ZYI = \gamma^2 I \qquad \gamma = \sqrt{ZY}$$

Both of these are linear second order differential equations whose solutions are $e^{\gamma x}$ and $e^{-\gamma x}$. The equation for the voltage is

$$V = V_1 e^{-\gamma x} + V_2 e^{\gamma x}$$

If we take the derivative of this equation we can find the solution for the current.

$$\frac{dV}{dx} = -V_1 \gamma e^{-\gamma x} + V_2 \gamma e^{\gamma x} = -IZ$$

$$I = \frac{\gamma}{Z}\left(V_1 e^{-\gamma x} - V_2 e^{\gamma x}\right)$$

We will consider only lossless lines. In that case R = 0 and G = 0.

$$ZY = -\omega^2 LC = \gamma^2$$

Then $\gamma x = j\omega\sqrt{LC}\, x = j\frac{\omega}{v}x$ The velocity of the waves is v

If we add back the $e^{j\omega t}$ term, then we have the following equation.

$$V = V_1 e^{j(\omega t - \beta x)} + V_2 e^{j(\omega t + \beta x)} \qquad \beta = \frac{\omega}{v} = \frac{2\pi}{\lambda}$$

In this case we can see the two propagating modes. These expressions are similiar to the electromagnetic waves.

$e^{j(\omega t - \beta x)}$ propagates in the positive X direction

$e^{j(\omega t + \beta x)}$ propagates in the negative X direction

Of course, we will drop the $e^{j\omega t}$ term again and the equation for the voltage is given in phasor notation.

$$V = V_1 e^{-j\beta x} + V_2 e^{j\beta x}$$

and the current is: $I = \frac{\gamma}{Z}\left(V_1 e^{-j\beta x} - V_2 e^{j\beta x}\right)$

We can define the quantity γ/Z as an admittance because it relates the voltage to the current. The corresponding impedance is:

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{L}{C}}$$

This is called the characteristic impedance of the transmission line.

Each propagating wave has a voltage and current associated with it whose ratio is equal to the characteristic impedance. To properly complete the solution of the linear differential equation, we have to apply the boundary conditions, i.e. the loads.

The waves move along the transmission lines at the velocity of light in the medium. This means that there is a restriction on the product of the inductance and capacitance per unit length.

$$\frac{1}{\sqrt{LC}} = \text{Velocity of Light} = V_L$$

To find the characteristic impedance of a transmission line, we usually only have to find either the capacitance or inductance per unit length, because the velocity of light is known on the structure. This is handy because the capacitance per unit length of many structures can be found in the literature of electrostatics. Take the two equations that relate the inductance and capacitance per unit length to the impedance and the velocity of light.

$$Z = \sqrt{\frac{L}{C}} \qquad v = \frac{1}{\sqrt{LC}}$$

Solve the second equation for $\sqrt{L}$ $\qquad \sqrt{L} = \frac{1}{v\sqrt{C}}$

and substitute the result into the first equation. $\qquad Z = \frac{1}{v \cdot C}$

Similiarly we can find the characteristic impedance from the inductance per unit length.

$$\sqrt{C} = \frac{1}{v\sqrt{L}} \qquad Z = vL$$

For an example let us consider a coax line. From electrostatics we find that the capacitance per unit length of a coaxial capacitor is

$$C = \frac{2\pi \epsilon}{\ln(b/a)} \qquad \ln - \text{NATURAL Log}$$

Where ϵ is the permittivity of the material between the cylinders and b is the outer radius and a is the inner radius. Because we only use the ratio of b to a, they can also be the diameters of the two cylinders. The velocity of light in the dielectric between the cylinders is

$$\frac{v_L}{\sqrt{\epsilon_r}}$$

Where V_L is the velocity of light in free space and $\sqrt{\epsilon_r}$ is the square root of the dielectric constant (or relative permittivity). The characteristic impedance is found from the above relation.

$$Z_0 = \frac{1}{vC} = \frac{\sqrt{\epsilon_r} \ln(b/a)}{v_L 2\pi \epsilon}$$

From the discussion of plane waves we can recall that the velocity of light in free space is given in terms of the permittivity and permeability of free space.

$$v_L = \frac{1}{\sqrt{\epsilon_o \mu_o}} \qquad \epsilon = \epsilon_o \epsilon_r$$

Substituting in the above equation we have:

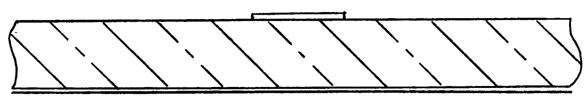
$$Z_o = \frac{\sqrt{\epsilon_r}\sqrt{\epsilon_o \mu_o}}{2\pi \epsilon_o \epsilon_r} \ln(b/a) = \sqrt{\frac{\mu_o}{\epsilon_o}} \frac{\ln(b/a)}{2\pi \epsilon_r}$$

Again for the discussion of plane waves we can recognize $\sqrt{\frac{\mu_o}{\epsilon_o}}$ as the intrinsic impedance of free space η = 376.73. The impedance of caox is given as

$$Z_o = \frac{\eta \ln(b/a)}{2\pi\sqrt{\epsilon_r}} = \frac{60}{\sqrt{\epsilon_r}} \ln(b/a)$$

This is the most used type of transmission line for microwave antennas.

When considering a transmission line that has a nonuniform dielectric, we cannot easily find the velocity of the wave. In this case we must find the inductance per unit length as well as the capacitance per unit length. For an example of this we will look at microstrip lines. A microstrip line is a flat metal strip on a dielectric sheet with a ground plane on the other side of the dielectric, see figure. For this line we can use



the following trick. We can find the capacitance per unit length when the dielectric is not present. For this case the characteristic impedance can be found because we know the velocity; it is just free space. Using this we can find the inductance per unit length.

$$L = Z_{oo}^2 C_o$$

Where Z_{oo} is the impedance of the microstrip line without the dielectric material. We can do this because the inductance per unit length is not effected by the dielectric constant. When we find the capacitance per unit length with the dielectric, we can find the characteristic impedance and the velocity of the waves on the transmission line.

$$Z_o = \sqrt{\frac{L}{C}} = Z_{oo}\sqrt{\frac{C_o}{C}}$$

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{Z_{oo}\sqrt{C_o C}}$$

The impedance of microstrip is found by numerical techniques. By using this method, the programmer only has to write a routine to find the capacitance per unit length for any dielectric slab and call it once with the dielectric constant and once with a dielectric constant of one. The two capacitances can be used to find the characteristic impedance and velocity of the waves on the transmission line.

TERMINATED TRANSMISSION LINE

When we solved the equations for the voltage and current on the transmission line, we obtained a solution which was two waves traveling in opposite directions on the line. The ratio of these two waves can be found from the boundary conditions imposed by terminations.

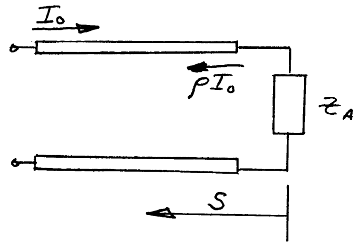
Suppose an antenna is connected to the end of a transmission line. Let Z_a be the impedance of the antenna. If a wave is sent down the transmission line towards the antenna, the ratio of the voltage across the line to the current flowing towards the antenna is given by the characteristic impedance of the line. When the wave reaches the antenna, the ratio of the voltage across the antenna terminals to the current flowing into the antenna is given by the input impedance of the antenna. It appears that there is a contradiction here because the transmission line demands that the ratio of the voltage to current must be equal to the characteristic impedance and on the other hand the antenna impedance also demands that the ratio of the voltage to current be equal to it. The only way both these conditions can be met is for a second wave to travel down the transmission line away from the antenna. We say that this wave has been reflected off the antenna.

Let us consider the equations for the current and voltage on the transmission line. It is convenient to measure distance from the antenna. Traditionally S is used for this distance. The voltage on the transmission line is expressed as:

$$V = V_0\left(e^{j\beta s} + \rho e^{-j\beta s}\right) \qquad \beta = \frac{2\pi}{\lambda}$$

Remember that $V_0 e^{j\beta s}$ is the wave traveling in the negative S direction or towards the antenna. $V_0 \rho e^{-j\beta s}$ is the reflected wave traveling away from the antenna (positive S direction). ρ is called the voltage reflection coefficient. We will take the reference direction for the current to be toward the antenna. The current is given by:

$$I = I_0\left(e^{j\beta s} - \rho e^{-j\beta s}\right)$$



The term $\rho e^{-j\beta s} I_o$ is negative because the current is flowing away from the antenna, see figure. This also comes out of the differential equation between the space derivative of the voltage and the current.

If we take the ratio of the voltage on the transmission line to the current on the transmission line, we will have the impedance looking back at the antenna through the transmission line. Remember we took the current toward the antenna as the reference which established the direction we are looking down the transmission line. In the

$$Z(s) = \frac{V}{I} = \frac{V_o(e^{j\beta s} + \rho e^{-j\beta s})}{I_o(e^{j\beta s} - \rho e^{-j\beta s})}$$

expression for impedance we recognize that $V_o/I_o = Z_o$, the characteristic impedance. If we also divide both the numerator and denominator by $e^{j\beta s}$, we have:

$$Z(s) = \frac{Z_o(1 + \rho e^{-j2\beta s})}{(1 - \rho e^{-j2\beta s})}$$

If we now look at the antenna terminal, S = 0, we have two expressions for the impedance which must be equal.

$$Z(0) = \frac{Z_o(1+\rho)}{1-\rho} = Z_A$$

We can use this to solve for the voltage reflection coefficient.

$$\rho = \frac{Z_A - Z_o}{Z_A + Z_o}$$

The important thing to notice here is that the impedance varies as we move along the transmission line away from the antenna. At any point along the transmission line we can consider this to be the impedance of the antenna and find the voltage reflection coefficient at this point. From the equation of the impedance along the transmission line we can identify a function of the reflection coefficient.

$$Z(s) = \frac{Z_o(1+\rho(s))}{(1-\rho(s))} \qquad \rho(s) = \rho_o e^{-j2\beta s}$$

Notice that the reflection coefficient phase varies at twice the rate as the traveling waves. The voltage reflection coefficient is the same as S_{11} of the scattering matrix.

Let us look at some common voltage reflection coefficients. If the antenna input impedance is the same as the transmission line characteristic impedance, we say the antenna is matched to the transmission line.

$$\rho = \frac{Z_o - Z_o}{Z_o + Z_o} = 0$$

If we have an open circuit, then the impedance is infinite. Divide the numerator and denominator of the expression for the voltage reflection coefficient by Z_A.

$$\rho = \frac{1 - Z_0/Z_A}{1 + Z_0/Z_A}$$

As $Z_A \rightarrow \infty$ in the limit $Z_0/Z_A \rightarrow 0$ and $\rho \rightarrow 1$.

Open circuit $\rho = 1 = e^{j0}$ magnitude 1, phase 0.

For a short circuit the impedance equals zero and the voltage reflection coefficient is

Short Circuit $\rho = \frac{-Z_0}{Z_0} = -1$ magnitude 1, phase 180^o.

VSWR

VSWR stands for the voltage standing wave ratio. Because there are two waves traveling in opposite directions on the transmission line, there will be points where the two voltages will add and points where the two voltages substract. It will appear that these voltages are stationary on the line even though they are the sum of two waves traveling at the speed of light in opposite directions. The ratio of the maximum voltage on the transmission line to the minimum voltage is VSWR.

The voltage along the transmission line is given as:

$$V = V_0(1 + \rho e^{-j2\beta s})$$

The maximum voltage occurs when the phase of $\rho(s)$ equals zero. The minimum voltage occurs when the phase of $\rho(s)$ equals 180^o.

$$\text{VSWR} = \frac{V_{MAX}}{V_{MIN}} = \frac{1 + |\rho|}{1 - |\rho|}$$

Likewise we can solve for the magnitude of the reflection coefficient.

$$|\rho| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1}$$

The VSWR is measured traditionally by inserting a small probe in the transmission line (a small antenna) with a detector connected to the probe. This probe was moved along the transmission line and the maximum and minimum voltages were recorded and the VSWR calculated from the ratio.

RETURN LOSS

We can express the magnitude of the voltage reflection coeffient in dB because it is the ratio of two voltages across the same impedance (the characteristic impedance of the transmission line). The reflection coefficient magnitude is always less than or equal to one which means the logarithm of the reflection coefficient magnitude will always be negative. To save ourselves from always having to write negative numbers we define the return loss

$$\text{Return Loss} = -20 \operatorname{Log} |\rho|$$

The return loss is so named because it is the ratio of the power reflected off the antenna to the power incident on the antenna.

The power ratio transmitted into the antenna is: $V_o I_o$

The power in the reflected wave is $V I^*$ using RMS values of current and voltage.

$$V I^* = V_o \rho(s)\, I_o \rho^*(s) = V_o I_o |\rho|^2$$

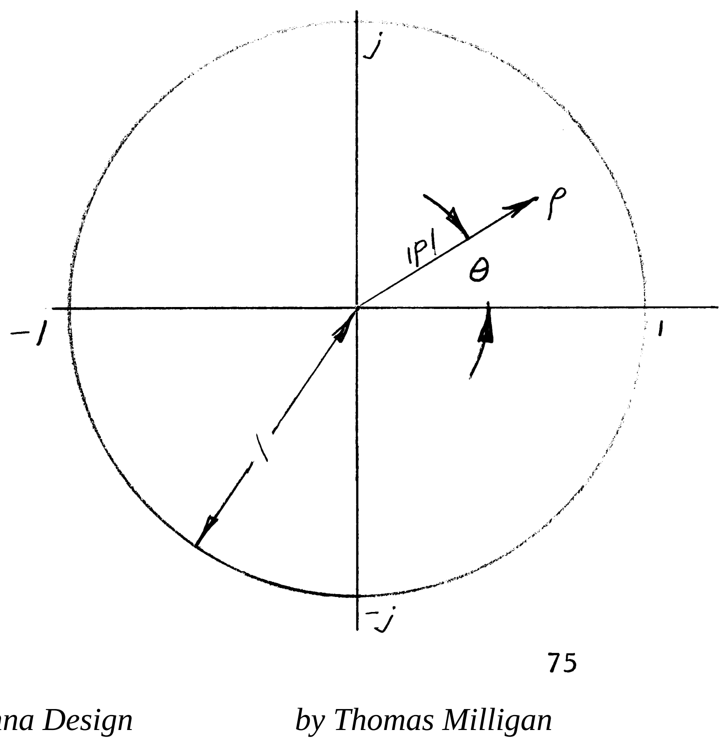
The power traveling toward the antenna is $V_o I_o$. By the conservation of energy the power delivered to the antenna must be the difference.

$$(1 - |\rho|^2)\, V_o I_o$$

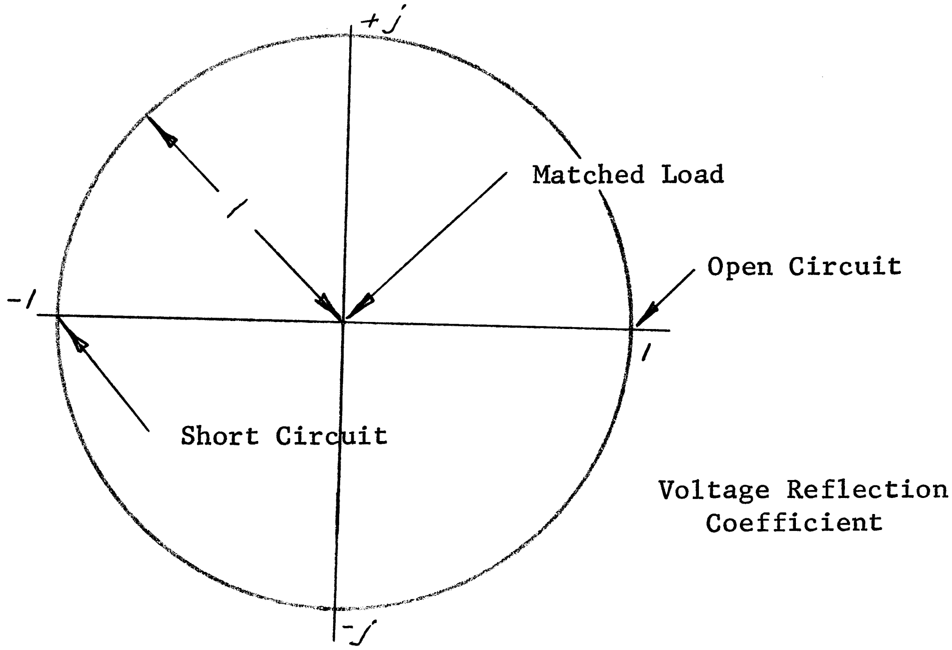
An alignment chart is given on page 76 of return loss versus VSWR and return loss versus transmission loss.

SMITH CHART

No study of transmission lines is complete without a discussion of the Smith chart. The voltage reflection coefficient is the ratio of two phasors which are the incident wave voltage and the reflected wave voltage. It can be represented as a magnitude and angle (phase) just like any other complex number. We can plot the reflection coefficient in polar coordinates; the maximum value is one for a passive component.



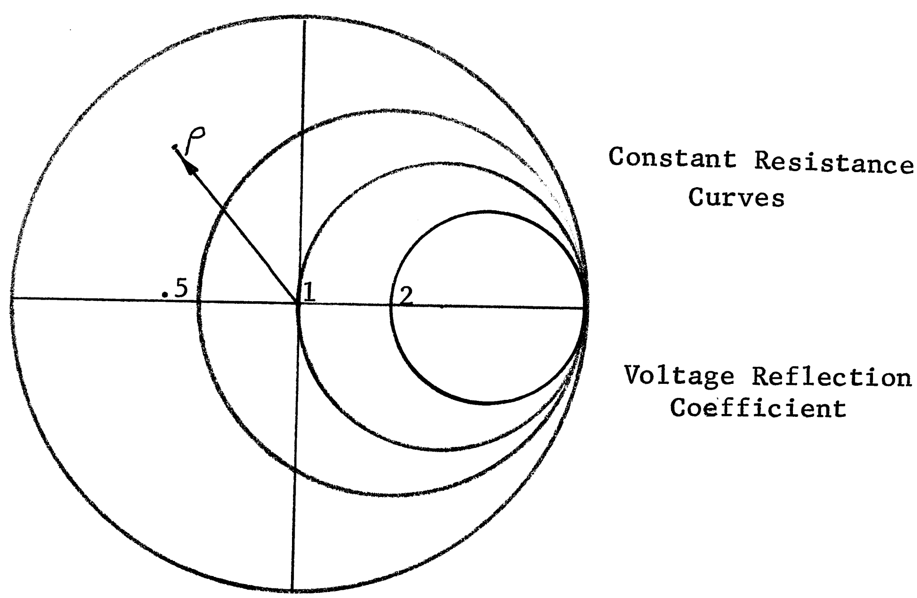
We can identify some of the common reflection coefficients.



Now that we identified a polar representation for the voltage reflection coefficient; P. H. Smith in 1939 overlayed on this a graph of the normalized impedance values with curves of constant resistance and reactance. The normalized impedance can be found from the voltage reflection coefficient.

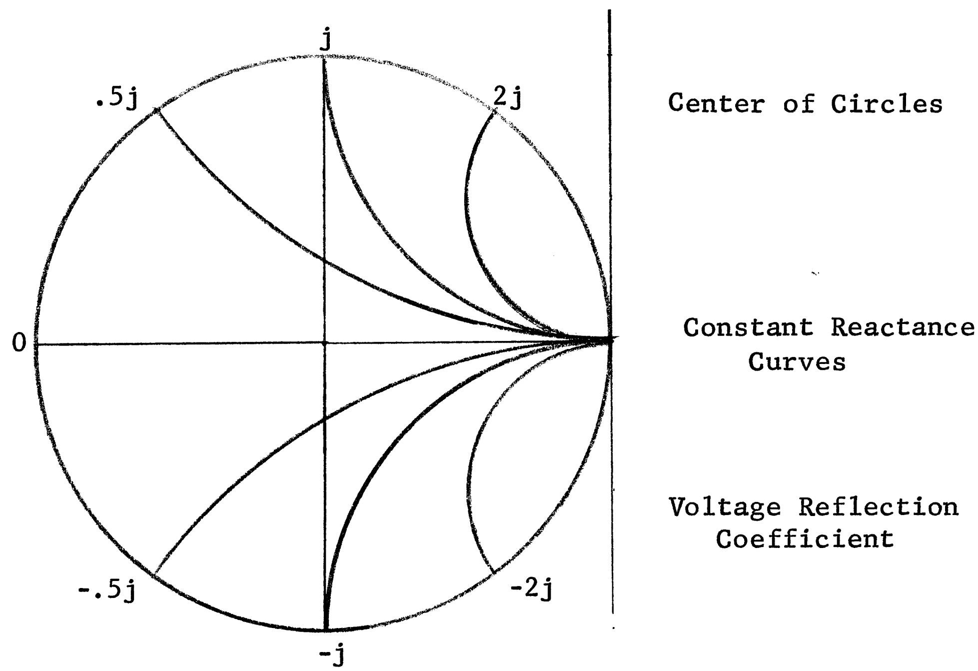
$$Z = \frac{Z_o(1+\rho)}{1-\rho}$$

The constant resistance curves are circles whose center lie on the zero angle line (real axis) and one point is at the open circuit point (reactance infinite). The normalized resistance of one passes through



the center of the outer one circle. Higher resistances are to the right of the origin (towards an open circuit) and smaller resistance circles cross the 180^{o} angle radius to the left of the origin (towards a short circuit).

The curves of constant reactance are also circles. These are centered on a line tangent to the one circle at the open circuit point. The X-axis



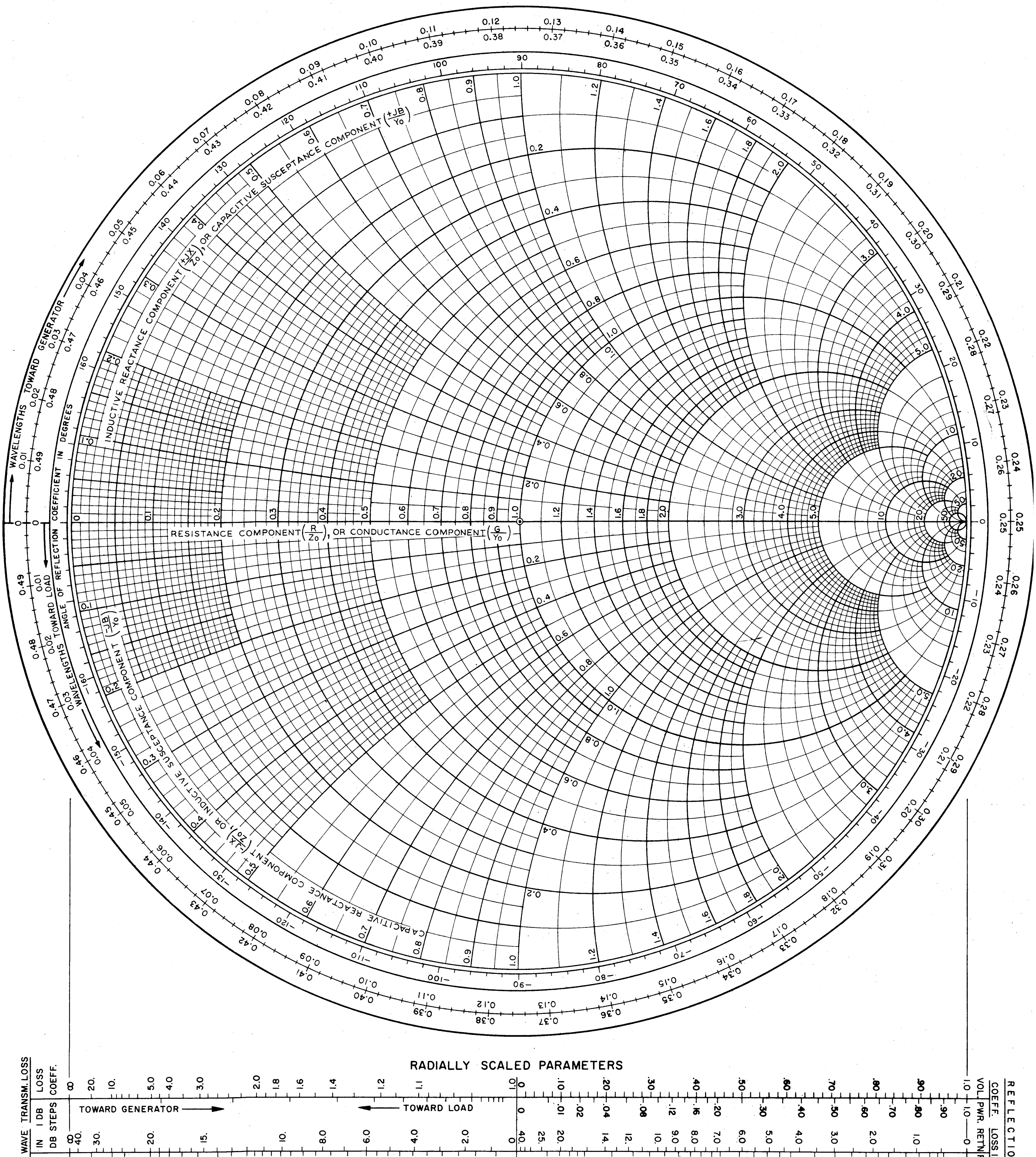
line corresponds to the zero reactance curve. Above the X-axis the reactance is positive or inductive; below it is capacitive (negative). The outer circle is zero resistance and corresponds to pure reactance.

When we combine the two sets of curves we have the full Smith chart given on page 79. Given the input impedance of an antenna we can locate the point on the Smith chart from this impedance normalized to the input transmission line. We can find the magnitude of the reflection coefficient as the distance from the center and the angle from the X axis. The angle of the reflection coefficient is scaled on the inner most circular scale. The magnitude of the reflection coefficient is found using the top most scale on the right side of the scale below the chart.

The VSWR can be found by drawing a cirle from the reflection coefficient point using the center of the chart as the center of the circle until it crosses the positive X-axis. The value on the X axis is the VSWR. The VSWR can also be found by using the scales at the bottom; it is given both in ratio and dB. The bottom scales also include the return loss and the transmission loss.

NAME	TITLE	DWG. NO.
Chapter 4 Transmission Lines		DATE
SMITH CHART FORM 5301-7560-N	GENERAL RADIO COMPANY, WEST CONCORD, MASSACHUSETTS	

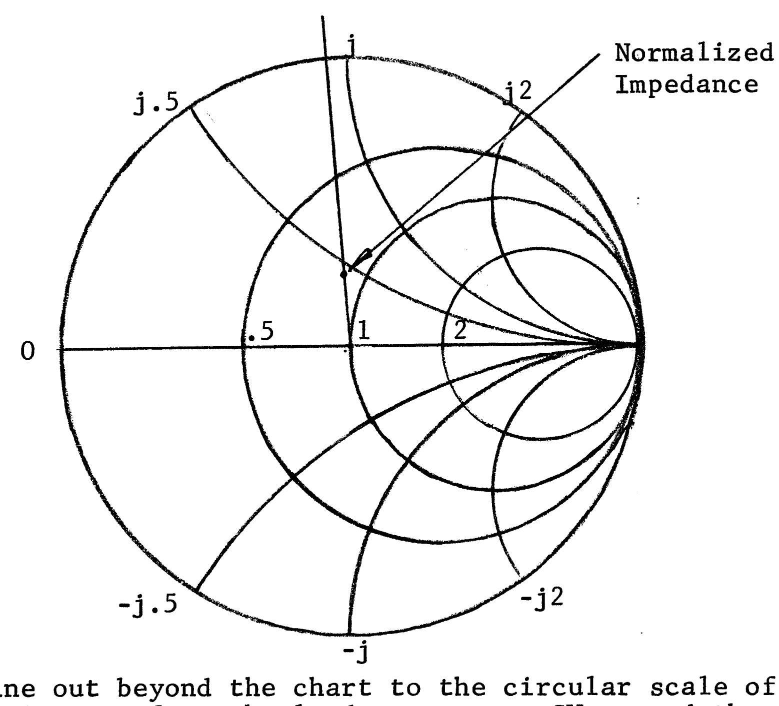
IMPEDANCE OR ADMITTANCE COORDINATES



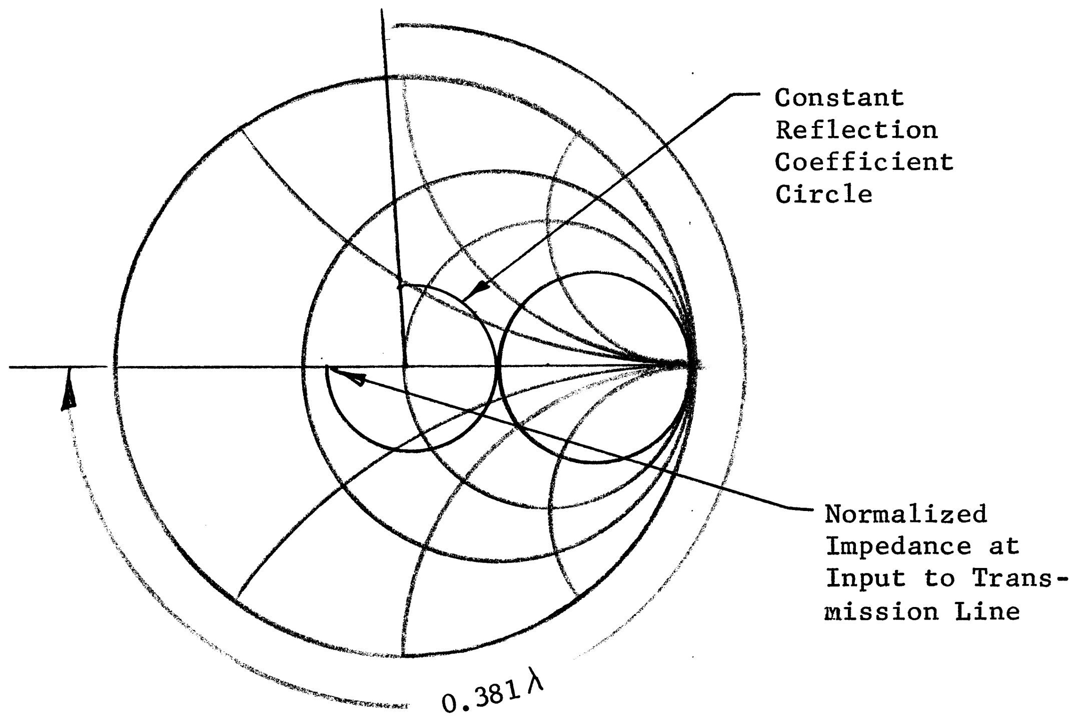
Electronics - Vol. 17, No. 1, PP. 130–133, 318–325, Jan. 1944

The Smith chart can be used to calculate the impedance along the transmission line. Remember that the distance S is measured away from the antenna and that the angle of the reflection coefficient decreases as the distance increases. Moving away from the antenna means decreasing angle or clockwise rotation on the Smith chart. The second point is that the reflection coefficient phase decreases at twice the rate as the two waves on the transmission line. One revolution of the chart corresponds to a half wavelength along the transmission line. The outer most circular scale is given in wavelengths. The magnitude of the reflection coefficient remains the same on a lossless transmission line. This corresponds to a circle centered on the origin of the Smith chart. To move an impedance down a transmission line, we find the normalized impedance on the chart, draw a cirle through this point, draw a line to the outer scale. This locates the point using the scales. Next we use the outer scale to move down the transmission line in wavelengths. Using the outer scale we draw a radial line through the point on the scale. Where the radial line passes through the constant reflection coefficient circle, we can read the input impedance at that point on the transmission line from the Smith chart.

For an example let us take an antenna with an input impedance of 73 + j42 and find the input impedance at the input of a transmission line .381 wavelengths long whose characteristic impedance is 86.5 ohms. First we must normalize the impedance of the antenna to the transmission line. Divide both the real and imaginary parts of the impedance by the characteristic impedance of the transmission line. The normalized impedance is 0.84 + j0.49. Plot this point on the Smith Chart.



Draw a radial line out beyond the chart to the circular scale of wavelengths. Since we are moving away from the load, we rotate CW around the chart on a constant reflection coefficient circle. After rotating clockwise on the

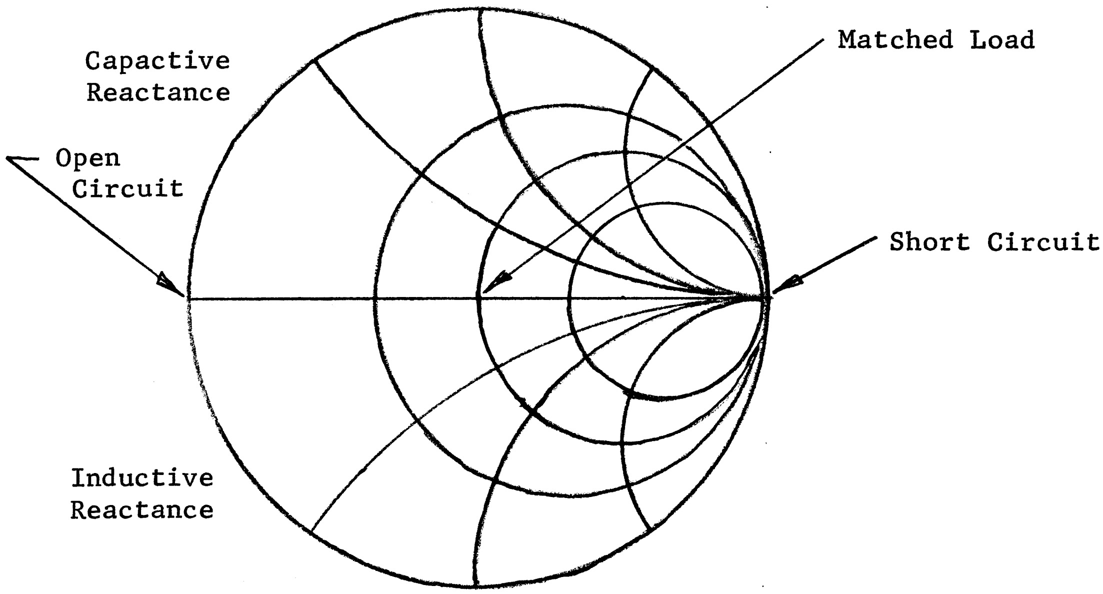


wavelength scale 0.381 wavelengths, draw a radial line from the wavelengths scale to the center of the chart. On the chart draw a circle of constant reflection coefficient through the normalized impedance. Where the radial line from the center of the chart to the wavelength scale and the circle of constant reflection coefficient intersect is the normalized impedance of the antenna looking into a section of transmission line. We read the normalized impedance to be 0.58 ohms. To unnormalize this impedance we multiply it by the characterististic impedance of the transmission line, 86.5 ohms. When we do this the result is 50.2 ohms. If at this point we would connect a 50 ohm transmission line, then the antenna would be matched to this tranmission line.

This example of matching an antenna was taken from the input impedance of a half-wave dipole antenna. In practice this would be a poor design because the reactive part of the impedance of the antenna could be reduced by shortening the antenna. When this is done the real part is also reduced. The second problem is using a long section of tranmission line to match the antenna. The electrical length of this line will change with frequency and the antenna will no longer be matched. Matching sections should be as short as possible to get the maximum bandwidth.

The Smith chart may be used with admittance as well as impedance. In this case we must add 180° to the angles of the reflection coefficient on the chart. After we have done this we can use the chart the same as before only we must use admittances instead of impedances. This is handy when we use shunt stubs to match the antenna.

We can identify some of the key points on the Smith chart when using it with admittances.



Smith Chart using Admittance Coordinates

Since to change from impedance to admittance we need only add 180° to the phase angle of the reflection coefficient on the Smith chart, we can use the chart to change impedances to admittances and vice versa. In fact when we are matching an antenna we can switch back on the chart depending on whether a shunt or series reactance is to be added to the admittance or impedance respectively. We need to keep track of whether we have impedance or admittance during the process.