ARRAYS

We will study arrays before considering individual antennas because an antenna can be analyzed as an array of point sources. It is a good topic to get a feel for the addition of the radiation from individual parts of an antenna. In fact we can roughly predict the patterns of some antennas by breaking up the antenna into small pieces and considering it as an array. We will start by considering only isotropic radiators. Later we will apply pattern multiplication when we use real antennas as elements of the array to get the patterns of these arrays. Last we will consider some array synthesis We must keep in mind that there are no real antennas which have an isotropic pattern. It has been proven that no finite sized antenna can have an isotropic pattern. If we look down on antennas like broadcast towers they are omnidirectional and are approximated well by isotropic antennas.

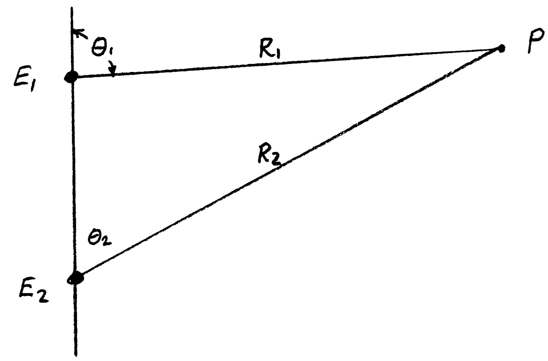
ADDITION OF FIELDS

Suppose there is a point which is receiving radiation from two different antennas which are radiating at the same frequency. How do we add these two radiations? The radiation from each antenna at the observation point is represented by an electric field which is a phasor quantity. The power is the product of two phasors. We know from circuit theory that we must add voltages or currents at a point in a circuit if the two are at the same frequency or correlated. These phasor voltages or currents can cancel each other out at a point in the circuit. So it is with an antenna. When dealing with uncorrelated signals, such as noise, we add the power in each signal. The answer is that we must add the electric (or magnetic) fields and not the power since the signals are highly correlated (same frequency). Since the magnetic field in an electromagnetic wave is proportional to the electric field, we can deal strictly with the electric fields and, as usual, ignore the magnetic field. Remember that the electric field is expressed as a phasor, and the real and imaginary parts of two electric fields must be added separately. The second point is that the electric field is a vector quantity. Each component must be added separately. Throughout this discussion we will assume in most cases that all the antennas have the same polarization. We can ignore the problem of adding antenna fields with different polarizations.

RADIATION APPROXIMATION

Suppose we have two antennas spaced some distance apart. If both antennas are radiating, then spherical waves are traveling away from each antenna. Far away from both antennas the ratio of the two distances from the antennas becomes nearly one. The electric field from each of them at the observation point becomes proportional only to the input power to each antenna and the average distance from them. But no matter how far away from the two antennas the observation point becomes, the difference in the distance to the two antennas is a constant. The phase difference between the two signals depends on the difference in distances.

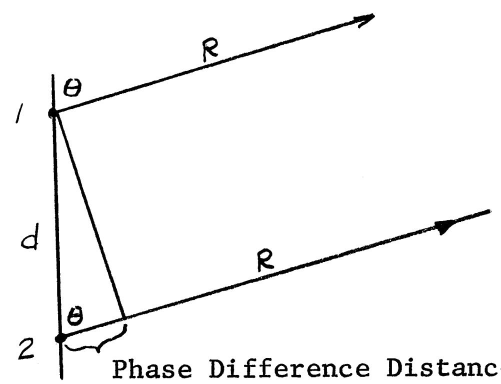
We cannot ignore the difference between the distances for phase angles of the electric field.



The point P in the figure above is an observation point. The field at this point when the two antennas are isotropic radiators is given by:

$$\frac{E_1}{R_1} e^{-j\beta R_1} + \frac{E_2}{R_2} e^{-j\beta R_2} \qquad \beta = \frac{2\pi}{\lambda}$$

Where E_1 and E_2 are the electric fields from each antenna at a unit distance. When the point P is moved farther and farther away, the two angles θ_1 and θ_2 become equal and $1/R_1$ and $1/R_2$ are considered equal. The phase difference distance is found from the triangle in the figure below.



Phase Difference Distance = d Cos θ

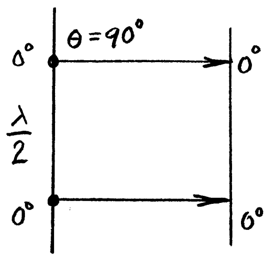
These are the radiation approximations.

ARRAYS OF TWO ANTENNAS

Assume that we have two isotropic antennas spaced on the Z axis. From symmetry we can see that the pattern will be the same for any spherical angle ϕ for a constant cone angle θ. If the antenna pair is rotated, there is no difference in the arrangement of the antennas. For the first case assume the antennas are spaced a half wavelength apart.

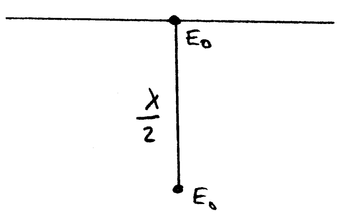
Case 1. The two antennas have the same amplitude and phase angle.

We can find the maximums and minimums of the pattern by a few simple arguments. Consider the pattern at $\theta = 90^{\circ}$. In the far field of the pattern we use the radiation approximation which says that the directions from the two antennas are the same. We can use any line parallel to the Z axis to add the fields.



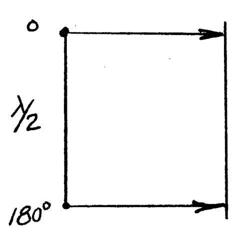
The two spherical waves travel the same distance to the line and the equal phases add. At this point we have a maximum of the pattern.

At $\theta = 0^{\circ}$ we can use a line perpendicular to the Z axis to find the amplitude of the pattern.

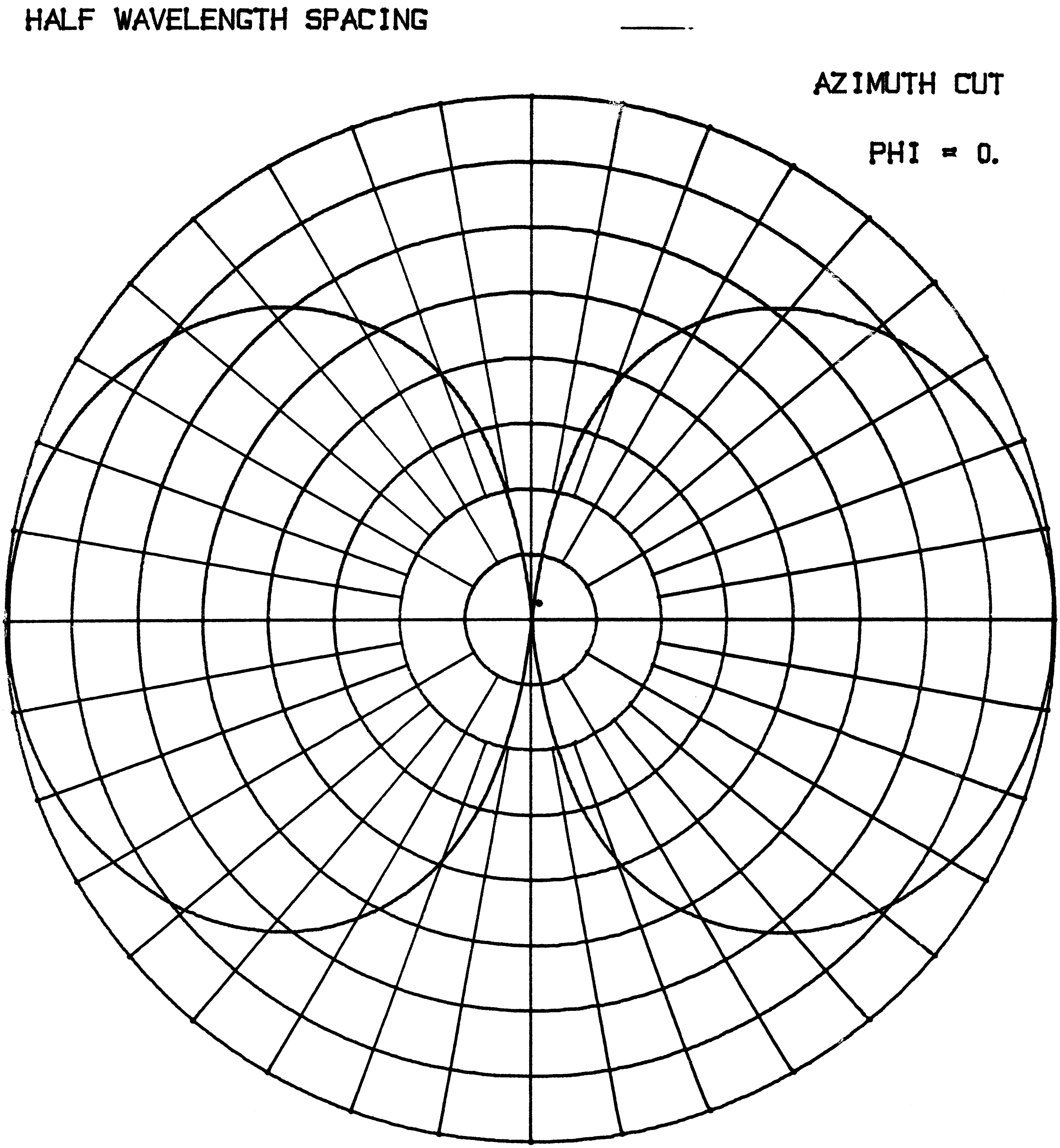


The phase of the electric field from antenna 1 is zero. The wave from antenna 2 travels a half wavelength before reaching the phase reference plane and has a phase angle of 180°. In the far field radiation approximation the distance is the same to both antennas for the amplitude. When we add these two electric fields, we get a null. By the same argument we can find a null at $\theta = 180^{\circ}$. But a better way to find this result is to notice that there is a plane of symmetry half way between the antennas. The pattern is the same on both sides of the plane. The full pattern is given on page 86. The maximums and minimums are exactly where we predicted them to be from these simple arguments.

Case 2. The antennas have the same amplitude but differ in phase by 180°.



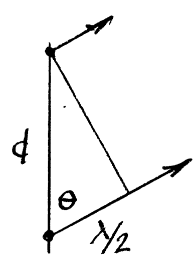
On the line parallel to the Z axis the two electric fields cancel since they are out of phase by 180°. On a line perpendicular to the Z axis through antenna one at the top, the phase from antenna 1 is zero. The wave from antenna 2 travels a half wavelength across the array and decreases the phase by 180° giving a resultant phase of zero from antenna 2 at the reference plane. These two electric fields add giving a maximum. There is a plane


HALF WAVELENGTH SPACING
AZIMUTH CUT
PHI = 0.

of odd symmetry half way between the two antennas. If we add 180^o to the phase of each antenna, we get the same problem we had above only turned over. The amplitude of the pattern at $\theta = 180^o$ is the same as the amplitude at $\theta = 0$. The phase of the pattern is 180^o with respect to the pattern at $\theta = 0$. The full pattern is given on page 88.

Case 3. Two antennas are spaced one wavelength with equal amplitude and phase.

On a line parallel to the axis between them the electric fields from the two antennas will add in phase and give a maximum. If we consider the plane perpendicular to the Z axis, the phase from the second antenna has decreased by 360^o to be in phase with antenna 1. There is a maximum at $\theta = 0^o$ and by symmetry at $\theta = 180^o$. Since it is impossible to have an isotropic antenna, there must be a null between the two maximums. If we pick a line where the difference in the distance between the two antennas is $\lambda/2$, then the difference in the phases of the waves from the two antennas will be 180^o and there will be a null in the pattern.

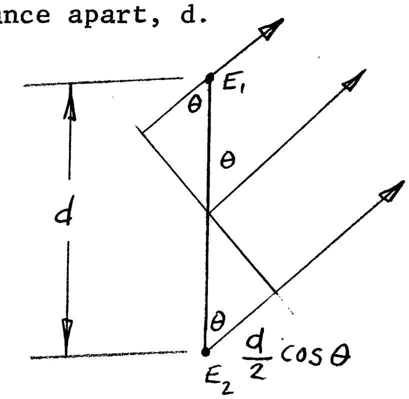


In the figure the difference in distance to the two points on the plane defining the direction of propagation is $\lambda/2$ and there is a null in the pattern. The distance d is one wavelength. The null is determined by triangle as:

$$\cos\theta = 1/2 \qquad \theta = \cos^{-1}(\tfrac{1}{2}) = 60^o$$

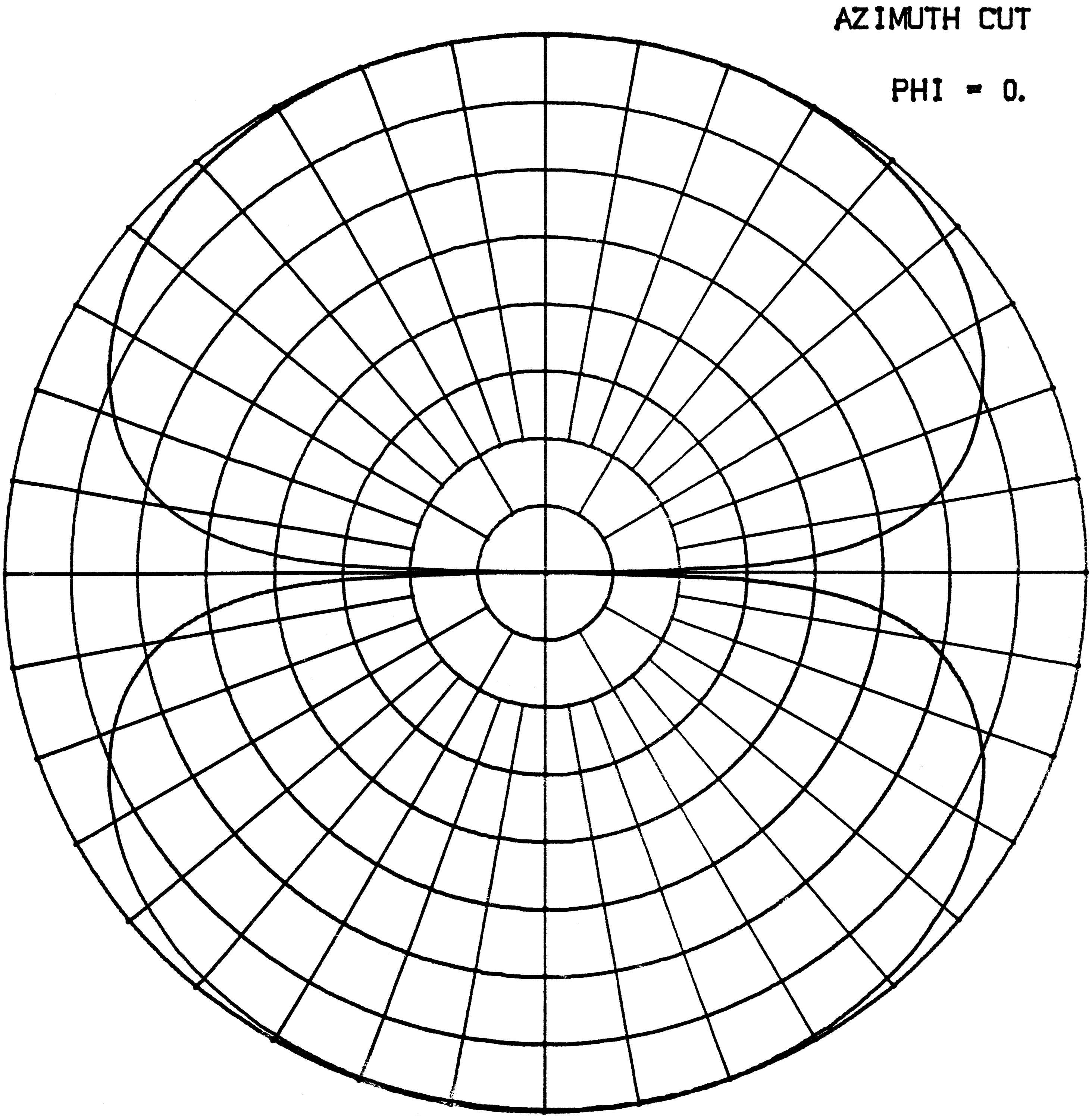
The nulls in the pattern are at 60^o. The full pattern is plotted on page 89.

For an isotropic array we just add the electric fields propagating from the antennas to get the electric field associated with the radiation intensity. We need to put this in formal mathematical terms. When we find the far field point, we can pick any plane parallel to the plane defined by the far field direction to add the electric fields. The electric field on the plane from an antenna is the wave propagated to the plane ignoring the 1/R dependence of the electric field. If S is the distance from the plane to the antenna in the direction of propagation, then the electric field at the plane is $E_o e^{j\beta S}$. Where E_o is the magnitude of the electric field at the far field point and β is the propagation constant $= 2\pi/\lambda$. Consider the two isotropic antennas spaced some distance apart, d.

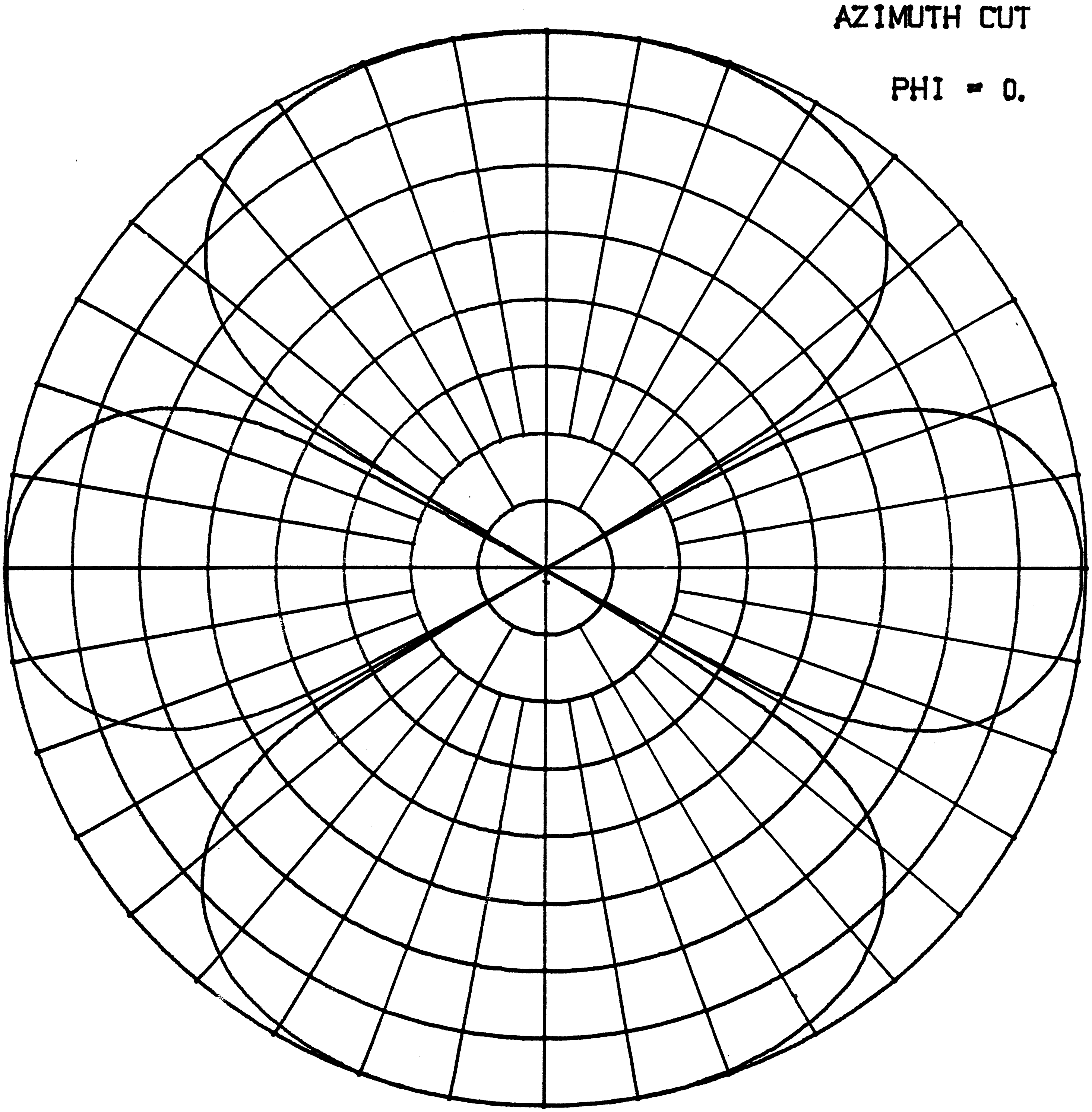


We will pick a plane half way between the antennas to be the reference plane. Given θ as the direction of radiation we can find the phase shift of each array element to the reference plane which is defined by the radius vector in the direction θ. Positive distance is in the direction of propagation. Remember that phase decreases with increasing distance. The distance from both antennas to the reference plane is the same: $d/2 \cos\theta$

HALF WAVELENGTH SPACING 0, 180 DEG PHASES



ONE WAVELENGTH SPACING



The field at the distant point of the far field is

$$E = E_1 e^{j\frac{\beta d}{2}\cos\theta} + E_2 e^{-j\frac{\beta d}{2}\cos\theta}$$

E_1 and E_2 are phasors which have magnitude and phase. In general we can find the field from any number of isotropic sources using this expression.

$$E = \sum_{i}^{N} E_i e^{j\beta d_i(\theta,\phi)}$$

Where E_i is the magnitude and phase (phasor) of the field from the i-th antenna at the far field point and $d_i(\theta,\phi)$ is the distance of the i-th antenna to the reference plane defined by the radius vector in the direction (θ,ϕ) of propagation.

Now consider the equal amplitude and phase case with two antennas.

$$E = E_0 \left(e^{j\frac{\beta d}{2}\cos\theta} + e^{-j\frac{\beta d}{2}\cos\theta} \right)$$

From the Euler identity: $\text{Cos } b = \frac{1}{2}(e^{jb} + e^{-jb})$, we can reduce this to

$$E = 2E_0 \cos\left(\frac{\beta d}{2}\cos\theta\right)$$

The maximum electric field from a pair of antennas is twice the electric field from one antenna. The radiation intensity is proportional to the square of the electric field or increased by a factor of four over the radiation from one of the antennas. But remember that the input power had to be split between two antennas. The net gain is two. The directivity is found by integrating the radiation intensity.

$$\text{Radiation Intensity} = 4\, E_0^2 \,\text{Cos}^2\left(\frac{\pi d}{\lambda}\,\text{Cos}\,\theta\right)$$

Since the pattern has symmetry around the Z axis, we need only find the average radiation intensity over the θ variable. The second point is the symmetry about the X-Y plane which says we need integrate only from 0 to $\pi/2$. If we note that the integral of the Sin θ from 0 to $\pi/2$ is one, then the average radiation intensity is found from the following integral.

$$U_{AV} = \int_0^{\pi/2} 4E_0^2 \cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right) \sin\theta \, d\theta$$

Remember that $\sin\theta \, d\phi$ is the differential length of the ϕ variable. We can make the following substitution and solve the integral.

$$a = \frac{\pi d}{\lambda}\cos\theta$$

$$da = -\frac{\pi d}{\lambda}\sin\theta \, d\theta$$

Substituting these into the integral it is reduced to

$$U_{AV} = \frac{\lambda}{\pi d}\int_0^{\frac{\pi d}{\lambda}} 4E_0^2 \cos^2 a \, da$$

$$= \frac{4E_0^2\lambda}{\pi d}\left[\frac{\pi d}{2\lambda} + \frac{1}{4}\sin\frac{2\pi d}{\lambda}\right]$$

$$= \frac{4E_0^2}{2}\left[1 + \frac{\lambda}{2\pi d}\sin\frac{2\pi d}{\lambda}\right]$$

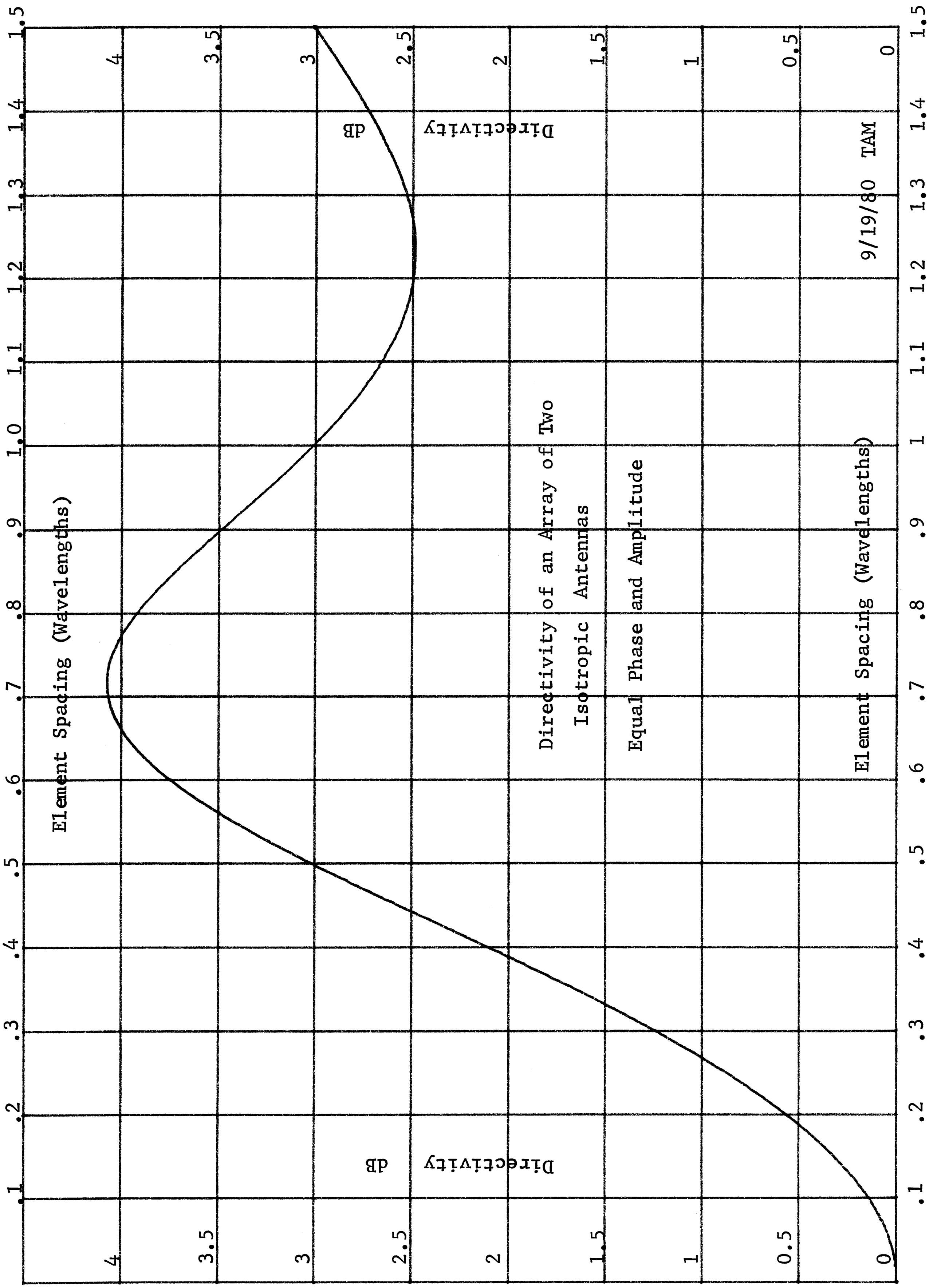
Using this result we can find the directivity.

$$\text{Directivity} = \frac{U_{MAX}}{U_{AV}} = \frac{2}{1 + \dfrac{\sin\left(\frac{2\pi d}{\lambda}\right)}{\frac{2\pi d}{\lambda}}}$$

The directivity is dependent on the distance between the antennas. For $\lambda/2$ the Sin X / X term is zero and the directivity is 2. The directivity peaks up for a distance between the antennas greater than $\lambda/2$. A graph of the directivity of an array of two isotropic radiators is given on page 92.

It appears like there is a contradiction here. Directivity is a measure of the increase in the radiation intensity over an isotropic radiator and if we assume 100% efficiency, then gain = directivity. The maximum radiation intensity from the two antennas is twice the radiation from the antennas considered one at a time. Since each antenna is an isotropic radiator, it would appear that the directivity should always be two. But from the calculation on the pattern, this is not so. What has happened?

Each antenna in the array receives radiation from the other antennas in the array. When this happens, the effective voltage on the antenna is not the same when it radiates by itself. If we consider the interactions between the two antennas, the problem becomes unduly complicated. When working with arrays, it is easier to find the directivity from the pattern. Otherwise for large arrays we would have to consider all the interactions between the pairs of antennas. When we calculate the input impedance of the antennas in the array, we must do just that. The input impedance of each antenna in the array then depends on the magnitude and phase of the voltage feeding the other antennas in the array.


Directivity of an Array of Two
Isotropic Antennas
Equal Phase and Amplitude
Element Spacing (Wavelengths)
Directivity dB
9/19/80 TAM

The expression for the two special cases of equal amplitude and phase arrays of two elements comes from the formula above.

Half-wave spacing $d = \lambda/2$

$$\text{Radiation Intensity} = U_{MAX} \cos^2\left(\frac{\pi}{2}\cos\theta\right)$$

Full wave spacing $d = \lambda$

$$\text{Radiation Intensity} = U_{MAX} \cos^2\left(\pi \cos\theta\right)$$

These patterns are plotted on pages 86 and 89.

OUT OF PHASE ELEMENTS

Suppose that the second antenna in the array is 180° out of phase with respect to the first. Then the electric field in the far field is:

$$E = E_0\left(e^{j\frac{\beta d}{2}\cos\theta} - e^{-j\frac{\beta d}{2}\cos\theta}\right)$$

Using the Euler identity: $\sin a = \frac{1}{2j}\left(e^{ja} - e^{-ja}\right)$

The pattern response is reduced to $E = j2E_0 \sin\left(\frac{\beta d}{2}\cos\theta\right)$

The phase of the far field is 90° out of phase with respect to the center point between the two antennas plus, of course, the propagation phase shift. This is the meaning of the j term in the expression above.

$$\text{Radiation Intensity} = 4E_0^2 \sin^2\left(\frac{\beta d}{2}\cos(\theta)\right)$$

The directivity can be found by integrating the radiation intensity to find the average radiation intensity. Again the magnitude of the radiation intensity has symmetry about the Z axis and about the X-Y plane. The integration is only over the θ variable from 0 to $\pi/2$.

$$U_{AV} = 4E_0^2\int_0^{\pi/2} \sin^2\left(\frac{\pi d}{\lambda}\cos\theta\right)\sin\theta\, d\theta$$

We make the same substitutions to reduce the integral: $a = \frac{\pi d}{\lambda}\cos\theta$

$$da = -\frac{\pi d}{\lambda}\sin\theta\, d\theta$$

$$U_{AV} = \frac{4\lambda}{\pi d}E_0^2\int_0^{\frac{\pi d}{\lambda}} \sin^2 a\, da = \frac{4E_0^2\lambda}{\pi d}\left[\frac{\pi d}{2\lambda} - \frac{1}{4}\sin\left(\frac{2\pi d}{\lambda}\right)\right]$$

The maximum radiation intensity $= 4E_0^2 \sin^2\frac{\pi d}{\lambda} \qquad d \le \lambda/2$

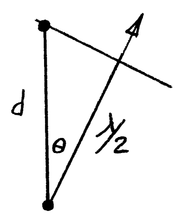
$$= 4E_0^2 \qquad d \ge \lambda/2$$

Using these expressions the directivity is:

$$\text{Directivity} = \frac{2\sin^2\left(\frac{\pi d}{\lambda}\right)}{1 - \frac{\sin\left(\frac{2\pi d}{\lambda}\right)}{\left(\frac{2\pi d}{\lambda}\right)}} \qquad d \le \lambda/2$$

$$\text{Directivity} = \frac{2}{1 - \frac{\sin\left(\frac{2\pi d}{\lambda}\right)}{\left(\frac{2\pi d}{\lambda}\right)}} \qquad d \ge \lambda/2$$

Beyond a distance of $d = \lambda/2$ the pattern no longer has a maximum at $\theta = 0$.



$$\cos\theta = \frac{\lambda}{2d}$$

$$\text{MAXIMUM } \theta = \cos^{-1}\left(\frac{\lambda}{2d}\right)$$

$$E_{MAX} = j2E_0 \sin\left(\frac{\pi d}{\lambda}\left(\frac{\lambda}{2d}\right)\right) = j2E_0\sin\left(\frac{\pi}{2}\right) = j2E_0$$

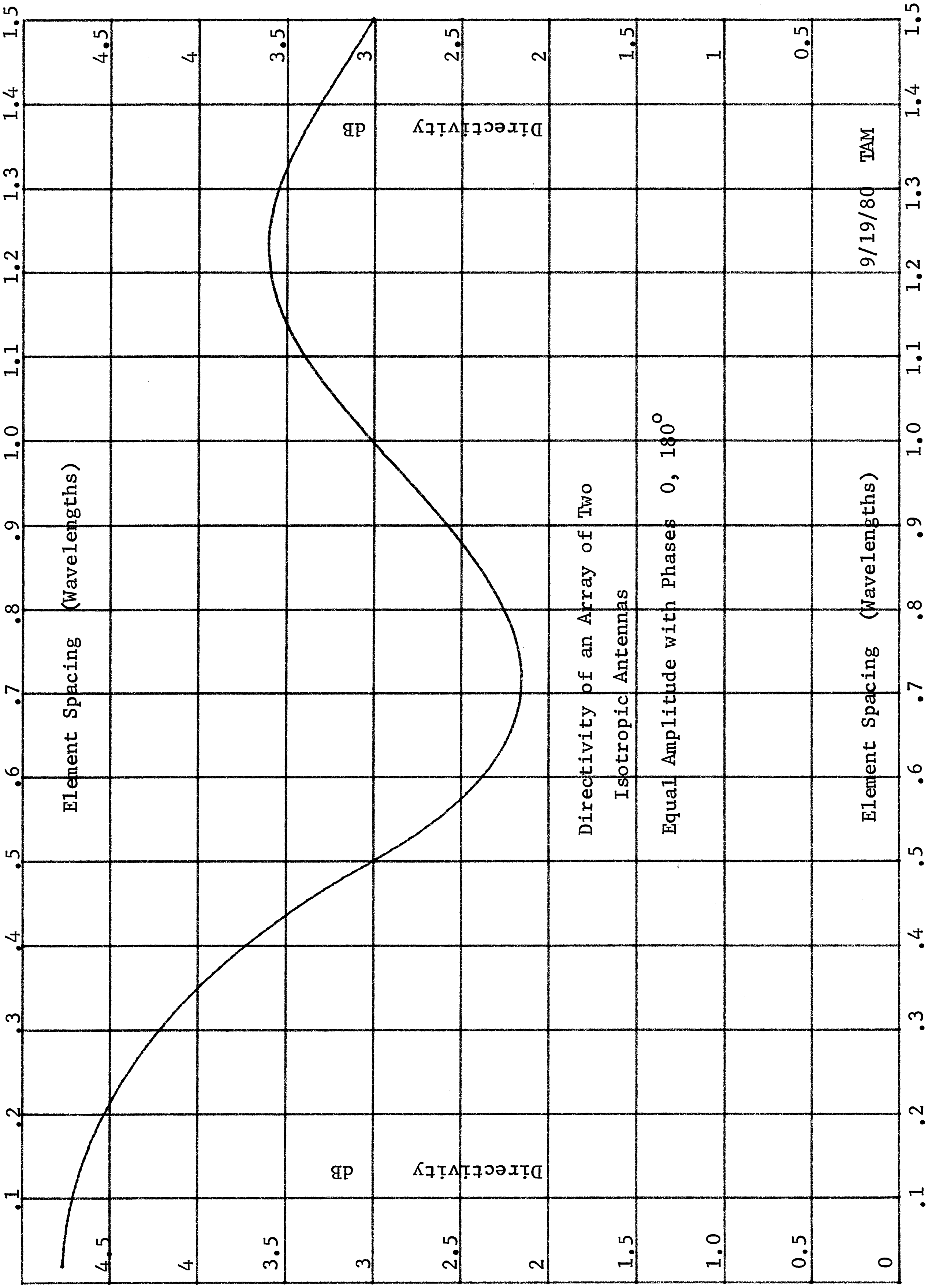
A curve of directivity versus spacing in wavelength is plotted on page 95. The interesting thing about this curve is that as the spacing between the two antennas becomes less than $\lambda/2$, the directivity increases. We can see this if we plot the pattern of an array of two antennas with phasing of 0 and 180° and a spacing of 0.2λ as on page 96. If we compare this pattern with the pattern on page 88, we see that it has s smaller beamwidth and higher directivity. At multiples of $\lambda/2$ the directivity is 3 dB or a ratio of 2.

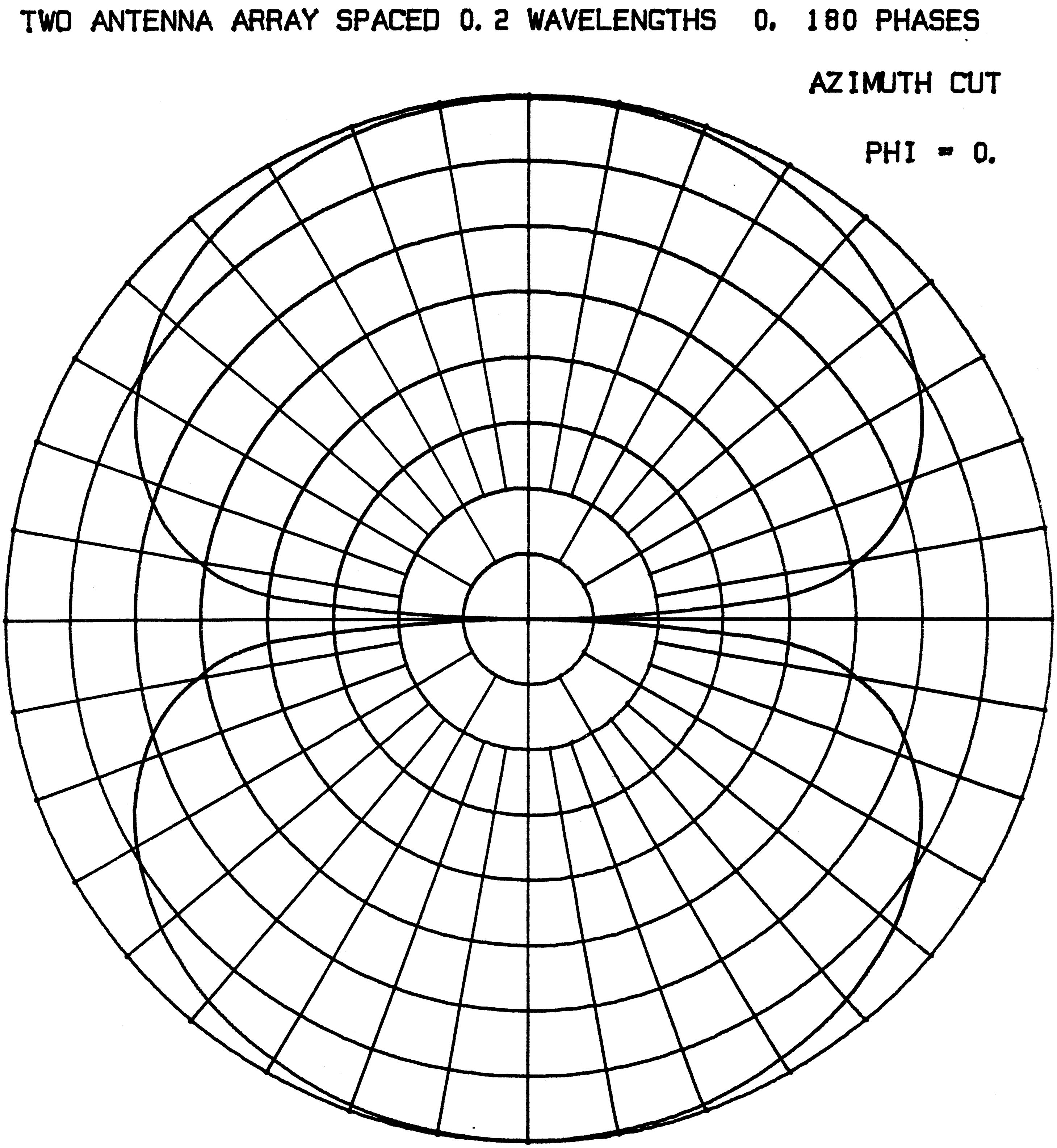
If we over lay the curve on page 92 where the two antennas are fed in phase with the curve on page 95 and convert to ratio we find that the average value of the two curves always equals two. These curves are given on page 97. We are now ready to talk about the interaction of the two antennas in an array.

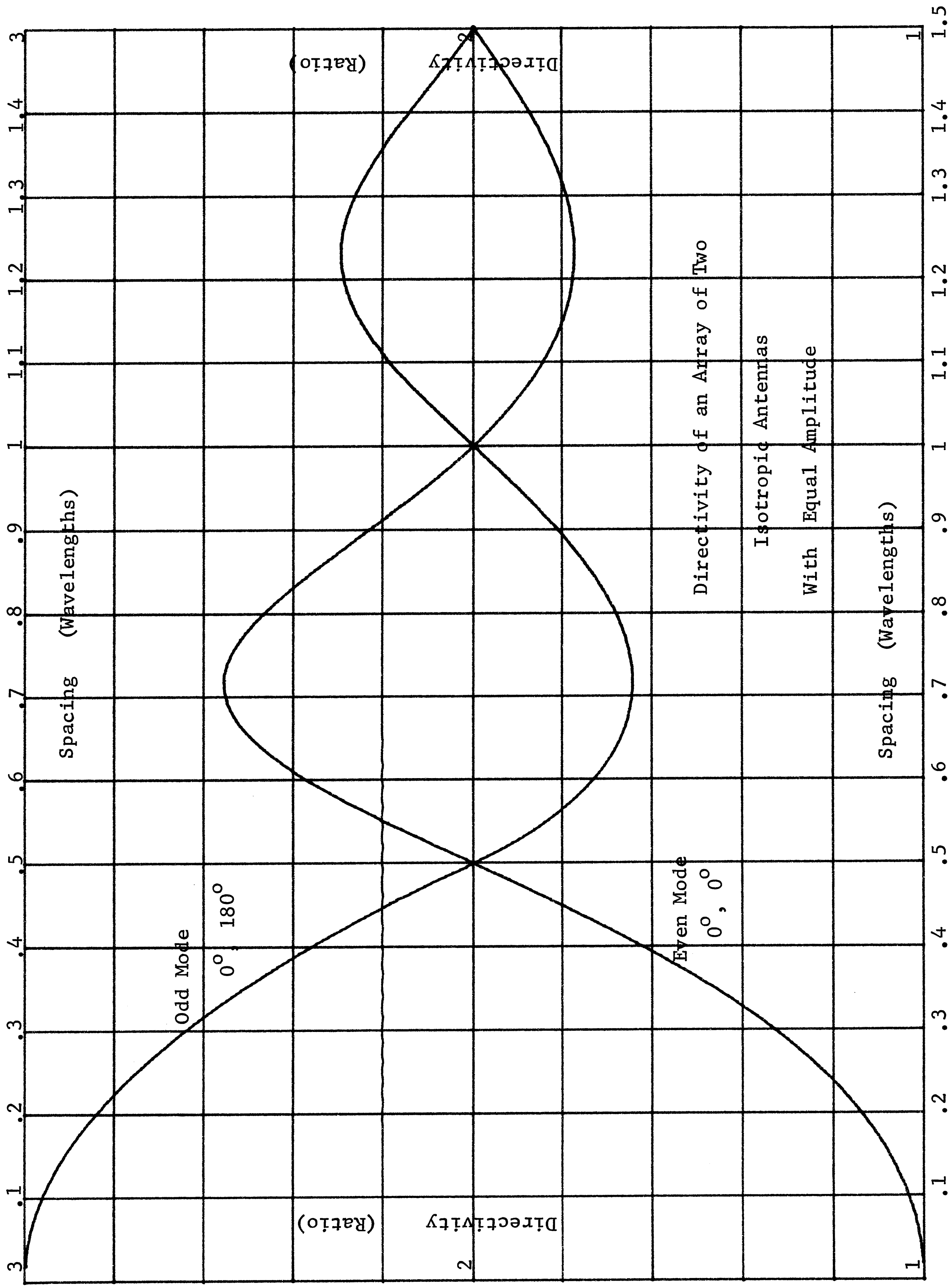
MUTUAL RESISTANCE

We can represent the circuit relationship of the two antennas as an impedance matrix for a two port device.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$


Directivity of an Array of Two
Isotropic Antennas
Equal Amplitude with Phases 0, 180°
Element Spacing (Wavelengths)
Directivity dB
9/19/80 TAM


TWO ANTENNA ARRAY SPACED 0.2 WAVELENGTHS 0, 180 PHASES
AZIMUTH CUT
PHI = 0.


Directivity of an Array of Two
Isotropic Antennas
With Equal Amplitude
Odd Mode
0°, 180°
Even Mode
0°, 0°
Spacing (Wavelengths)
Spacing (Wavelengths)
Directivity (Ratio)
Directivity (Ratio)
.1 .2 .3 .4 .5 .6 .7 .8 .9 1 1.1 1.2 1.3 1.4 1.5
1 2 3

We did this before when we discussed reciprocity. Using the reciprocity theorem we established that $Z_{12} = Z_{21}$ for antennas made of linear isotropic materials. Z_{11} is the self impedance of the antenna when no other antennas are present. Since both antennas are identical, $Z_{22} = Z_{11}$. The impedance matrix is reduced to:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{11} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

If we feed both antennas equally, then the matrix becomes:

$$\begin{bmatrix} V_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{11} \end{bmatrix} \begin{bmatrix} I_1 \\ I_1 \end{bmatrix}$$

$$V_1 = (Z_{11} + Z_{12}) I_1$$

The input impedance is $Z_{11} + Z_{12}$. This excitation is called the even mode.

When the two antennas are fed 180^{o} out of phase, the impedance matrix becomes:

$$\begin{bmatrix} V_1 \\ -V_1 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{11} \end{bmatrix} \begin{bmatrix} I_1 \\ -I_1 \end{bmatrix}$$

The input impedance is $Z_{11} - Z_{12}$ and is called the odd mode impedance. We can find the input power to one of the antennas as:

$$P_e = (R_{11} + R_{12})\, I_1^2 \qquad \text{even mode}$$

$$P_o = (R_{11} - R_{12})\, I_1^2 \qquad \text{odd mode}$$

Where R_{ij} is the real part of the impedances. We have another calculation of the power radiated from the calculation of the directivity. The integral formula for directivity is:

$$\text{Directivity} = \frac{4\pi U_{MAX}}{\iint U \sin\theta \, d\phi \, d\theta} = \frac{4\pi U_{MAX}}{P}$$

The surface integral of the radiation intensity in the denominator is the input power. For the even mode array the integral is:

$$\iint U \sin\theta \, d\phi \, d\theta = 4\pi E_o^2 \left(2 \left(1 + \frac{\lambda}{2\pi d} \sin\left(\frac{2\pi d}{\lambda}\right)\right)\right)$$

This is the sum of the power into both antennas. The power into one of these antennas is just ½ of this.

$$P_e = 4\pi E_o^2 \left(1 + \frac{\lambda}{2\pi d} \sin \frac{2\pi \lambda}{d} \right)$$

If we had only one isotropic radiator, the power into it is the surface integral of the radiation intensity.

$$P_0 = 4\pi E_0^2 = R_{11} I_1^2 \qquad (I,\ \mathrm{RMS})$$

In the even mode the two expressions for the input power can be equated.

$$4\pi E_0^2\left(1 + \frac{\lambda}{2\pi d}\sin\frac{2\pi d}{\lambda}\right) = R_{11} I_1^2\left(1 + \frac{R_{12}}{R_{11}}\right)$$

Using the equations above we can identify the R_{12}/R_{11} term.

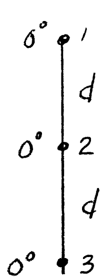
$$\frac{R_{12}}{R_{11}} = \frac{\lambda}{2\pi d}\sin\frac{2\pi d}{\lambda}$$

This is the normalized mutual impedance of an isotropic radiator. We can find the input power to one of the antennas in the odd mode excitation:

$$P_{ODD} = 4\pi E_0^2\left(1 - \frac{R_{12}}{R_{11}}\right)$$

When we use this with the maximum radiation intensity, we have the same formula for the directivity of the odd mode as when it was calculated using the integral. This also points out one of the problems of the odd mode excitation. The input resistance which is the radiation resistance becomes smaller and smaller as the two antennas are brought closer together. The antenna material losses become larger with respect to the radiation resistance and the efficiency decreases. The input impedance becomes so low that it is hard to match the antenna.

We can use the method of mutual resistance to calculate the directivity of a three element array with the elements uniformly fed and spaced.



We can find the maximum radiation intensity as $9E_0^2$ from an axis parallel to the Z axis. To find the directivity the total input power to all the elements must be calculated. The power into elements 1 and 3 is the same and is given by:

$$P_1 = 4\pi E_0^2\left(1 + \frac{R_{12}(d)}{R_{11}} + \frac{R_{12}(2d)}{R_{11}}\right)$$

The power into the center element is:

$$P_2 = 4\pi E_0^2\left(1 + \frac{2R_{12}(d)}{R_{11}}\right)$$

It couples equally to both antennas on either side. The total power into the array is the sum of these powers.

$$P_T = 2P_1 + P_2 = 4\pi E_0^2\left(3 + \frac{4R_{12}(d)}{R_{11}} + \frac{2R_{12}(2d)}{R_{11}}\right)$$

The directivity is found by:

$$\mathrm{Directivity} = \frac{4\pi U_{MAX}}{P_T}$$

Substituting the expression for the total power and the maximum radiation intensity, the directivity is found from the formula above.

$$\text{Directivity} = \frac{9}{\left(3 + 4\frac{R_{12}(d)}{R_{11}} + 2\frac{R_{12}(2d)}{R_{11}}\right)}$$

One of the important thing to notice about the three element array is that the input resistance of all three antennas is not the same. From symmetry we can see that the two end elements have the same input resistance; they see the same enviroment. The middle element sees another enviroment. The input resistance for the three antennas is given by

$$R_1 = R_3 = R_{11}\left(1 + \frac{R_{12}(d)}{R_{11}} + \frac{R_{12}(2d)}{R_{11}}\right) \qquad R_2 = R_{11}\left(1 + \frac{2R_{12}(d)}{R_{11}}\right)$$

If we look at the expression for the mutual resistance of two isotropic radiators, then we will see that the mutual resistance is zero for a spacing of $\lambda/2$. In that case the input resistance is the same for all the antennas. As the array becomes very large, the center elements begin to all have about the same input resistance. In general the input impedance of an antenna in an array is dependent on the excitations of all the other antennas in the array.

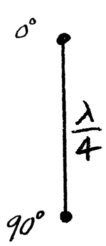
If we know the normalized mutual resistance formula for a real antenna pair, then we can use the same formula to find the directivity. The nice part is that the integral of the radiation intensity does not have to be found. For the general equally spaced broadside array the directivity is given by

$$\text{Directivity} = \frac{N^2}{N + 2\sum_{k}^{N-1}(N-k)\frac{R_{12}(kd)}{R_{11}}}$$

This has been plotted on page 101 using the normalized isotropic mutual resistance. For a real antenna array this curve would be slightly different because it depends on the mutual impedance expression. This has been done for the case of half wave dipole when the elements are parallel and has been plotted on page 102. We can notice a few changes from the case of the isotropic antenna arrays. The increase in directivity is (N = 2) more than 3 dB at $\lambda/2$ spacing. This is because the mutual resistance is not zero at this spacing but is aready negative. For all the curves the maximum is higher than the isotropic case. Notice too that for N = 2, the maximum occurs for a smaller spacing 0.65λ versus 0.7λ. This second curve corresponds to half wave broadcast towers.

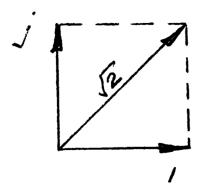
ENDFIRE ARRAY

Two isotropic antennas are spaced $\lambda/4$ on the Z axis with a 90° phase difference between them. Let us first find the maximums and minimums.

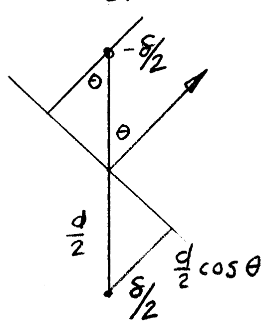


Draw a plane through the upper antenna and perpendicular to the Z axis. The signal from the lower antenna travels $\lambda/4$ and its phase is decreased by 90°. Both signals are at zero phase and add to a maximum. There is no angle which can be added to the phase of both antennas to give the same problem; there is no plane of symmetry. Draw a line through the

the bottom antenna perpendicular to the Z axis. The signal from the upper antenna travels $\lambda/4$ and its phase is decreased by 90°. The two signals on the plane have phases of 90° and -90°. Since the signals are 180° out of phase they cancel and there is a null in the pattern. Now consider a line parallel to the Z axis. Each antenna retains its phase. The two signals are out of phase by 90°. We can draw a vector diagram in polar coordinates of the phasors to find the magnitude of the response. Compare this signal with the maximum signal of 2. The relative magnitude is: $\sqrt{2}/2 = 1/\sqrt{2}$. The power response is the square of this: 1/2. At $\theta = 90^\circ$ the pattern has its 3 dB point. The 3 dB beamwidth is 180°.



Instead of calculating the pattern of this particular pair, the general case of two antennas in an array with equal amplitude and an arbitrary phase difference will be solved. Consider two antennas spaced some distance d apart and a phase angle difference between them of δ. Assume that the upper one on the Z axis has phase $-\delta/2$ and the lower one a phase of $\delta/2$. The field is given by:



$$E_0\left(e^{j\left(\frac{\beta d}{2}\cos\theta - \delta/2\right)} + e^{-j\left(\frac{\beta d}{2}\cos\theta - \delta/2\right)}\right)$$

$$= 2E_0 \cos\left(\frac{\pi d}{\lambda}\cos\theta - \delta/2\right)$$

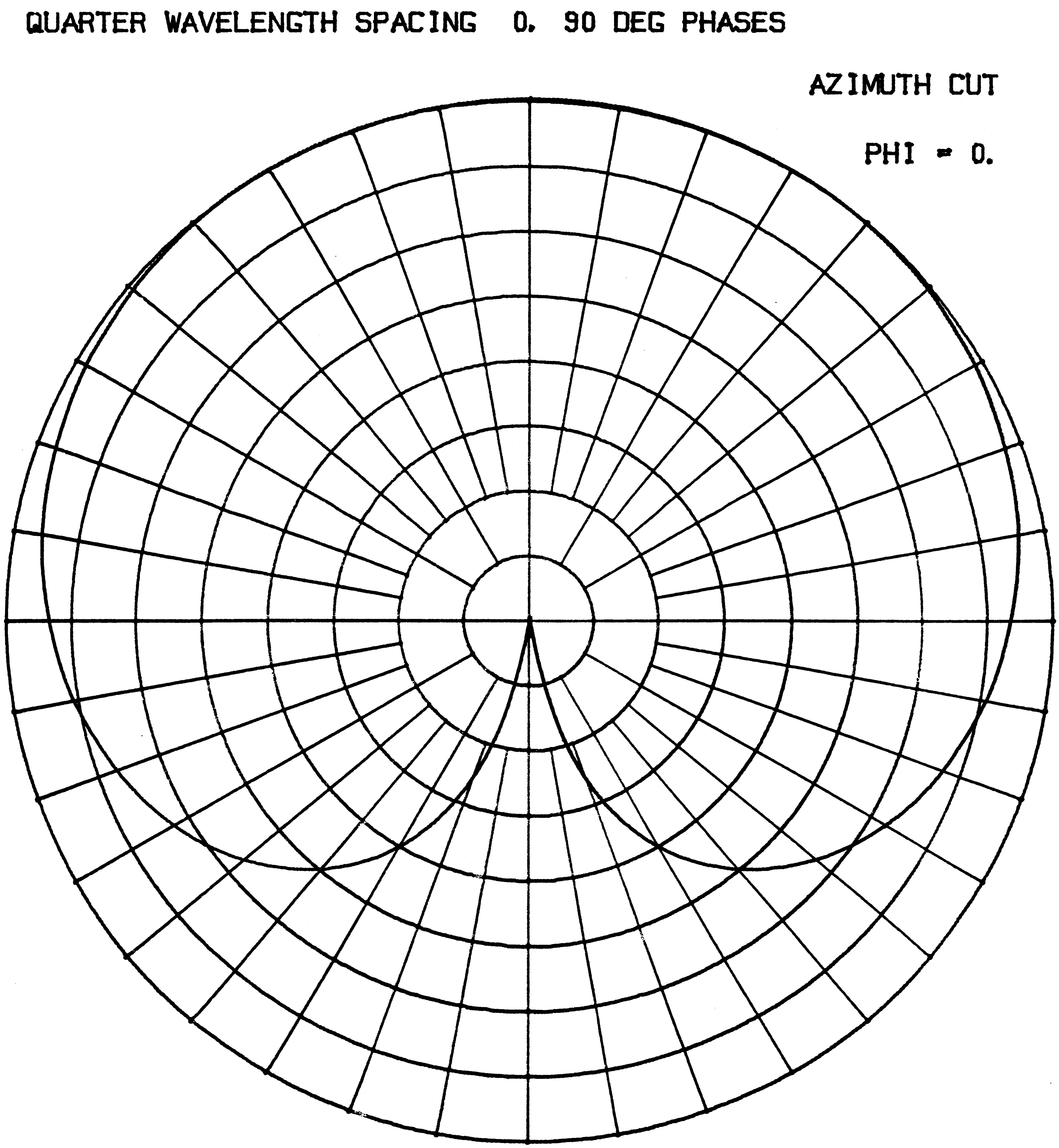
The maximums are at $\frac{\pi d}{\lambda}\cos\theta - \delta/2 = n\pi$

The minimums are at $\frac{\pi d}{\lambda}\cos\theta - \delta/2 = (2n-1)\frac{\pi}{2}$

This function squared can be integrated to find the average radiation intensity and the directivity. The result is:

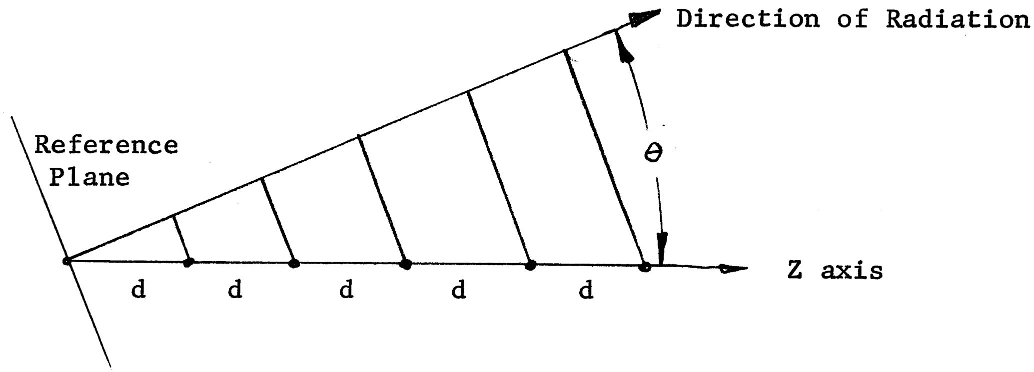
$$\text{Directivity} = \frac{2A_{MAX}}{1 + \dfrac{\sin\left(\frac{2\pi d}{\lambda}\right)\cos\delta}{\left(\frac{2\pi d}{\lambda}\right)}} \qquad A = \cos\left(\frac{\pi d}{\lambda}\cos\theta - \delta/2\right)$$

For the end fire case $\delta = \pi/2$ and $d = \lambda/4$; $\cos\delta = 0$ and $A_{MAX} = 1$. The directivity = 2. If $d \geq \lambda/4$, then $A_{MAX} = 1$. The directivity = 2 for all $d \geq \lambda/4$ if $\delta = 90^\circ$. The directivity of a two element end fire array equals 2 if $d \geq /4$; it is a constant. But for $d > \lambda/4$, the maximums of the beam will not occur at $\theta = 0$ but at $\theta = \cos^{-1}(\lambda/(4d))$ The pattern of the two element end fire array is plotted on page 104.


QUARTER WAVELENGTH SPACING 0, 90 DEG PHASES
AZIMUTH CUT
PHI = 0.

LINEAR ARRAY OF N ELEMENTS

Suppose there are N isotropic radiators equally spaced along a line and fed with equal amplitudes. In general we can assume a fixed phase shift between elements. This enables us to cover both broadside and endfire arrays.



The electric field can be referenced to the first (lowest) element on the Z axis. Let $\psi = \beta d \cos\theta + \delta$, where δ is the fixed phase shift from element to element. The electric field is given by

$$E = E_0\left(1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \dots + e^{j(N-1)\psi}\right)$$

Multiply this by $e^{j\psi}$

$$E e^{j\psi} = E_0\left(e^{j\psi} + e^{j2\psi} + e^{j3\psi} + e^{j4\psi} + \dots + e^{jN\psi}\right)$$

Substract these two equations; the result is

$$E\left(1 - e^{j\psi}\right) = E_0\left(1 - e^{jn\psi}\right)$$

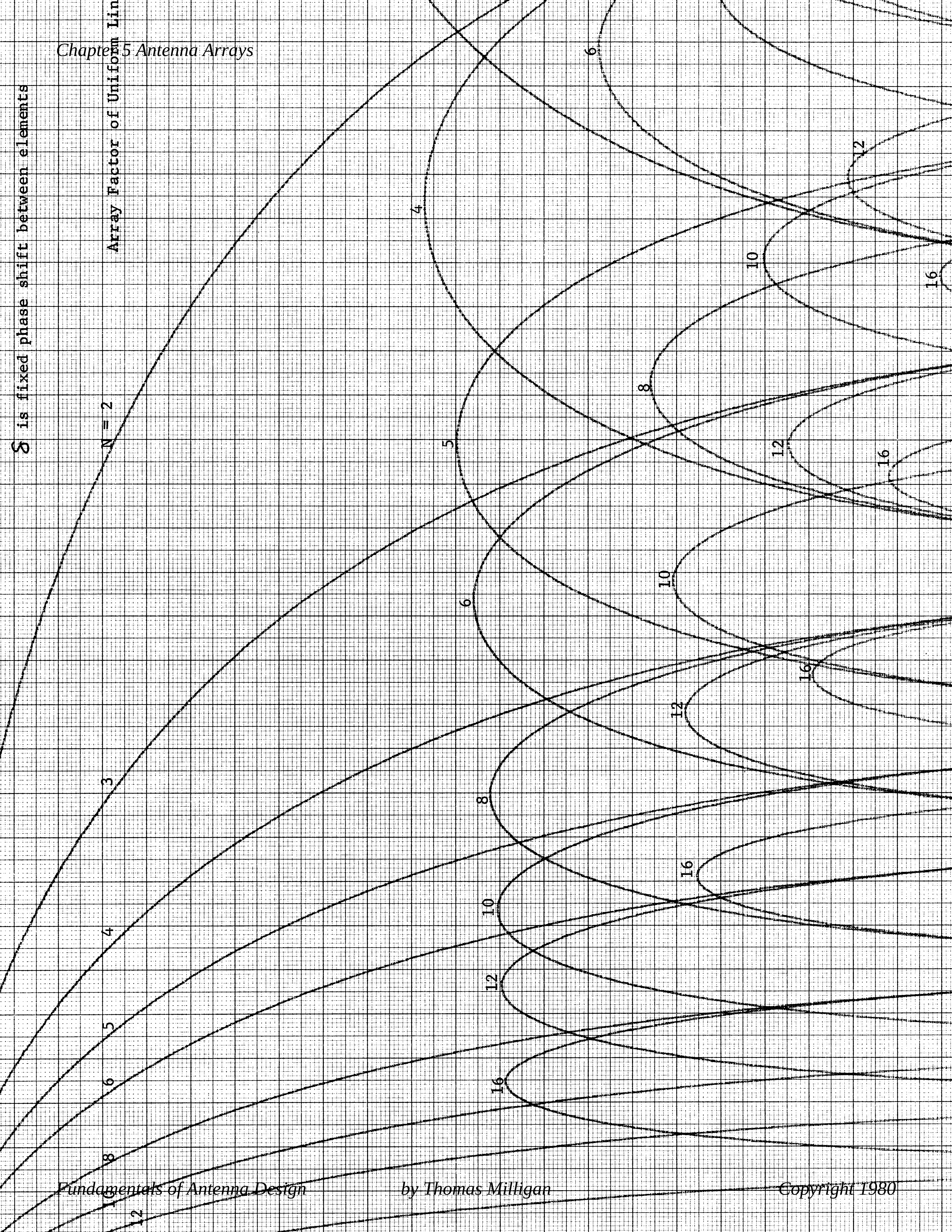
$$E = \frac{E_0\left(1 - e^{jn\psi}\right)}{1 - e^{j\psi}} = \frac{E_0\, e^{-j\frac{N\psi}{2}}\left(e^{j\frac{N\psi}{2}} - e^{-j\frac{N\psi}{2}}\right)}{e^{-j\psi/2}\left(e^{j\psi/2} - e^{-j\psi/2}\right)}$$

$$E = E_0 \frac{\sin(N\psi/2)}{\sin(\psi/2)} \angle \xi$$

If the center of the array is taken as the phase reference, then $\xi = 0$. The array factor reduces to

$$\frac{\sin(N\psi/2)}{\sin(\psi/2)}$$

The maximum electric field is N and will occur when $\psi = 0$. A universal radiation curve is plotted on page 106 for this array. The abscissa is the factor $\psi = 2\pi d/\lambda \cos\theta + \delta$. From this plot the 3 dB beamwidths can be found. Note the curve repeats for negative values of ψ.


δ is fixed phase shift between elements
Array Factor of Uniform Lin
N = 2
3
4
5
6
8
10
12
16

Case 1. Broadside $\delta = 0$ $\beta d \operatorname{Cos}\theta = 0 \Rightarrow \theta = 90^{\circ}$

Case 2. Endfire $\theta = 0$ $\beta d + \delta = 0$ $\delta = -\beta d$

The beam may be pointed off axis as well. The beam will be at θ_1, then $\beta d \operatorname{Cos}\theta_1 + \delta = 0$ or $\delta = -\beta d \operatorname{Cos}\theta_1$.
Example: Suppose the beam will be pointed off at $\theta = 60^{\circ}$ and the elements are spaced $\lambda/2$. Then

$$\delta = -\frac{2\pi}{\lambda}\frac{\lambda}{2}\cos 60^{\circ} = -90^{\circ}$$

This array for four elements is plotted on page 108.

HANSEN AND WOODYARD END FIRE ARRAY

Hansen and Woodyard showed in 1938 that the directivity of an endfire array can be increased if the phase between the elements is changed to

$$\delta = -\beta d - \pi/N$$

Where N is the number of elements in the array. Take the case of 8 elements spaced at $\lambda/4$ intervals. The standard endfire array with 90° phase shift between elements is plotted on page 109. The Hansen and Woodyard endfire array has 112.5° phase shift between elements and has been plotted on page 110.

Phase Shift	3 dB Beamwidth	Directivity
-90°	80 degrees	9 dB
-112.5°	44	11.5

The beamwidth has been reduced and the directivity increased 2.5 dB. We have paid for the higher directivity with higher sidelobes. They increased from 13 dB to 9 dB.

PATTERN MULTIPLICATION

Up to this point we have only considered arrays of isotropic radiators. We will consider arrays made from identical elements which are pointing in the same direction. The general expression for the field from page 90 with nonisotropic antenna elements is

$$E = \sum_{i}^{N} E_o(\theta,\phi)\, E_i\, e^{j\beta d_i(\theta,\phi)}$$

Where $E_o(\theta,\phi)$ is the normalized pattern of the antenna in the array. We can separate this term out of the summation since each term is multiplied by it.

$$E = E_o(\theta,\phi)\sum_{i}^{N} E_i\, e^{j\beta d_i(\theta,\phi)}$$

POLAR DIAGRAM

Graphs such as the one on page 106 can be used to draw polar patterns of an array for various element spacings and interelement phasing by a graphical technique. We will call these polar diagrams.

On page 107b is an example of a polar diagram. The top of the page has a rectangular plot which is a repeat of the graph on page 106 for six elements. Notice that the abscissa is proportional to Cos θ. A polar diagram or pattern constructed from the top curve is drawn below. The maximum value of Cos θ is one when $\theta = 0$, which means that the maximum value of the abscissa is $\beta d + \delta$. The spacing between elements is d and the phase shift between elements is δ. In the example $\delta = 0$. The total range of the abscissa in the top curve is $2\beta d$. Around the polar pattern below is a dashed circle with a radius equal to βd. When the edges of the dashed circle are marked off on the upper curve, these lines are the limits of the visible region. No portion of the upper curve beyond the visible region will appear in the polar pattern since Cos θ is less than or equal to one.

Consider the vertical line drawn from the null at 60° in the top rectangular plot to the circle below. The distance from the center of the rectangular plot is $\beta d \cos 60^\circ$; this is also the distance on the polar diagram below using the dashed curve. When θ is measured from the horizontal line as shown, then a line drawn from the intersection of the vertical line and the circle to the center of the polar diagram will give a line on which the polar pattern amplitude will be the same as the rectangular plot above. Both points are in a null. If we consider the second vertical line which is drawn to the left of the origin, then we see that it intersects the dashed circle in two points. Draw radial lines from these points to the center of the polar diagram. The amplitude along this line will be equal to the amplitude on the rectangular curve. To construct the total polar pattern, we must consider many such vertical lines in the visible region, find the intersections, draw radial lines, and match amplitudes. In these diagrams the radius of the polar pattern is equal to the maximum amplitude of the rectangular plot.

The element spacing in the example on page 107b is 0.25 wavelengths. The maximum value on the rectangular plot still within the visible region is 90° for $\beta d \cos \theta$. On page 107c is polar diagram of the same number of elements only the element spacing has doubled to 0.5 wavelengths. The rectangular curve on this page is the same as the previous page. Now the dashed circle of the polar diagram will double its diameter from the previous example as shown. The maximum value of $\beta d \cos \theta$ in the visible region has expanded to 180°. As before we draw vertical lines from the rectangular plot on the top of the page to the dashed circle. At the intersection points we draw radial lines to the center of the diagram and mark off the radial distance the same as the vertical distance on the rectangular plot. Notice that in all these diagrams the curve for negative θ is the same as positive θ. The array has an axis of symmetry about $\theta = 0$ which means the pattern is the same for all values of ϕ for a given θ.

Let us now consider a case where there is a phase shift between each element of the array. The polar diagram on page 107e is such a case. We still have the rectangular plot of a six element uniform array drawn on the top of the diagram. The center of the polar diagram dashed circle has been displaced from the center of the rectangular plot by the value of the phase shift between elements, δ. The values of the abscissa on the rectangular plot in the visible region now range from $-\beta d + \delta$ to $\beta d + \delta$ and the dashed circle bounding the visible region has been moved off center.

In the example on page 107e the phase shift between elements is 90° and the spacing between the elements is 0.25 wavelengths. The polar pattern shows that this an endfire array with the peak of the beam at $\theta = 180^{\circ}$ which is found from the vertical line drawn from the center of the rectangular plot to the edge of the dashed circle below. The polar pattern is still constructed by drawing vertical lines from the rectangular plot to the dashed circle and marking amplitudes along the radial lines to the intersections to be the same as the rectangular plot. We could get a beam in $\theta = 0$ direction by using a phase shift between elements of -90°; then the dashed circle which bounds the visible region would be moved to the left of the center of the rectangular plot.

Let us consider the polar diagram of a Hansen and Woodyard endfire array. On page 107f is such a diagram for the same six element uniform array. The spacing between elements is a quarter wavelength which will establish the required phase shift between elements as $\beta d + 180/N$ or 120° for a six element array. The center of the dashed curve has been moved to 120° with respect to the center of the rectangular plot. Notice that the peak value of the rectangular plot is no longer in the visible region. To get a normal plot it will be necessary to scale the lengths off the rectangular plot so that the peak of the polar pattern will occur on the outside circle of the polar chart. We can see again that the beamwidth of the array has decreased from the beamwidth given on page 107c for the ordinary endfire array, but the sidelobes have increased levels. They have increased because the peak of the beam does not occur at the peak of the rectangular plot and the sidelobe levels are increased in the same proportion.

LINEAR PHASED ARRAYS

The polar diagram is a convenient way of looking at linear phased arrays. In a phased array the direction of the beam is controlled by the phase shift between the elements. Once we have picked the element spacing, we establish the diameter of the dashed circle which bounds the visible region of the pattern. As we vary the phase shift between elements we move the dashed circle to new points below the rectangular plot where we can construct the polar pattern.

The polar diagram on page 107g is one such case. The spacing between elements is 0.5 wavelengths and the phase shift between elements is 90°. We are still using a six element uniform array. The range of the abscissa has been doubled from the previous plots and shows that the rectangular plot is a periodic function. It has a period equal to 360°. The diameter of the dashed circle has been scaled to the abscissa of the rectangular plot so it is one-half the diameter of the plot on page 107c which has the same element spacing. The peak of the beam occurs when $\beta d \cos\theta + \delta$ equals zero.

$$\theta_{max} = \cos^{-1} \frac{\delta}{\beta d}$$

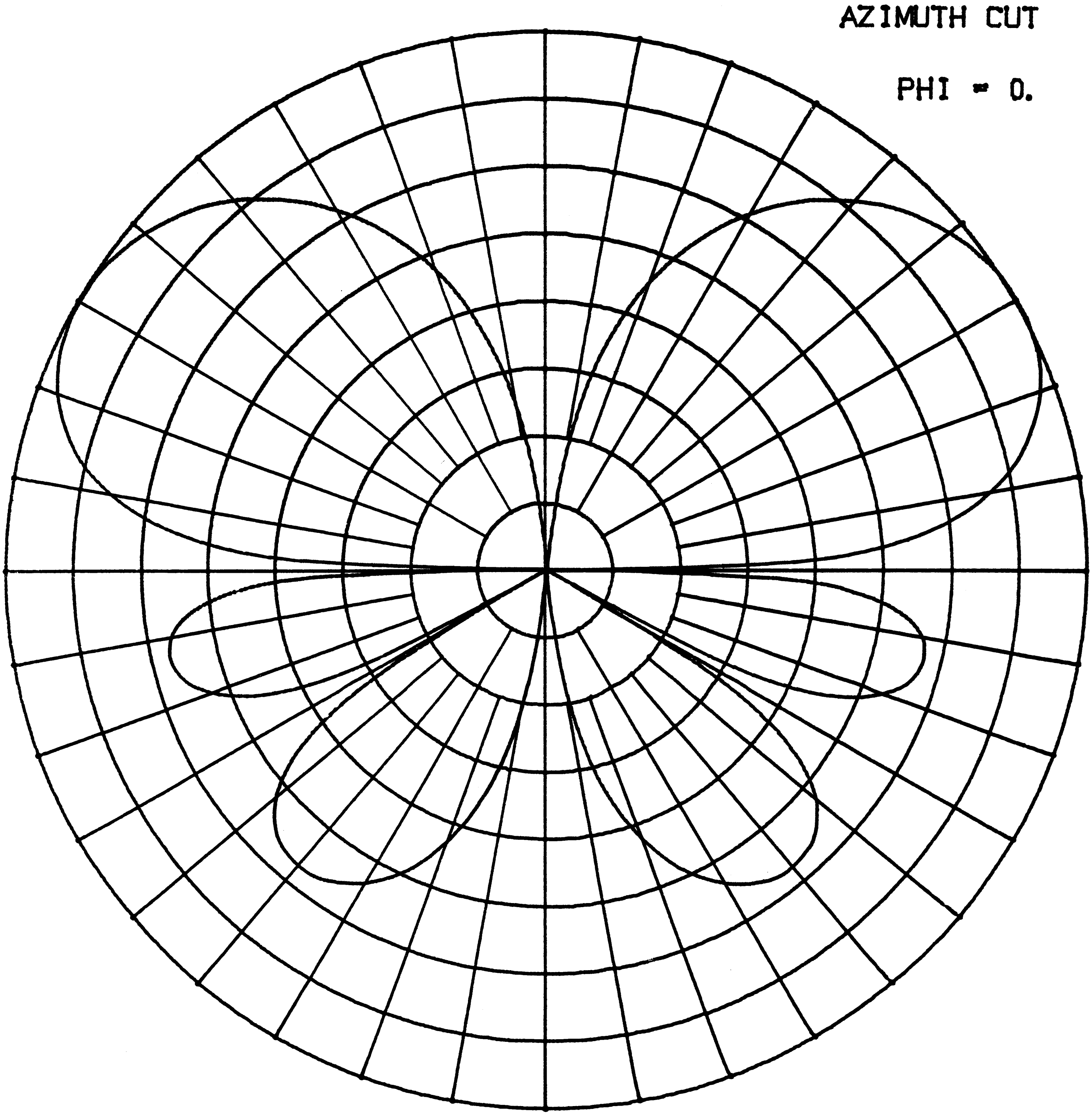
The peak of the beam on the polar pattern on page 107g is at 120^o or the beam is scanned 30^o off broadside.

The polar diagram on page 107i is the same array scanned further off broadside. The phase shift between elements is 150^o which gives a beam which is scanned 56.4^o off broadside. Notice that the beam for negative θ is starting to merge with the beam for positive θ into a single endfire beam. At the same time the sidelobe at $\theta = 0$ is growing. If we mentally shift the dashed circle over slightly to a center at 180^o, we can see that an endfire beam will be formed at $\theta = 180^o$ but that the beam at $\theta = 0^o$ will grow to the same amplitude. This is similiar to the two element array pattern on page 88.

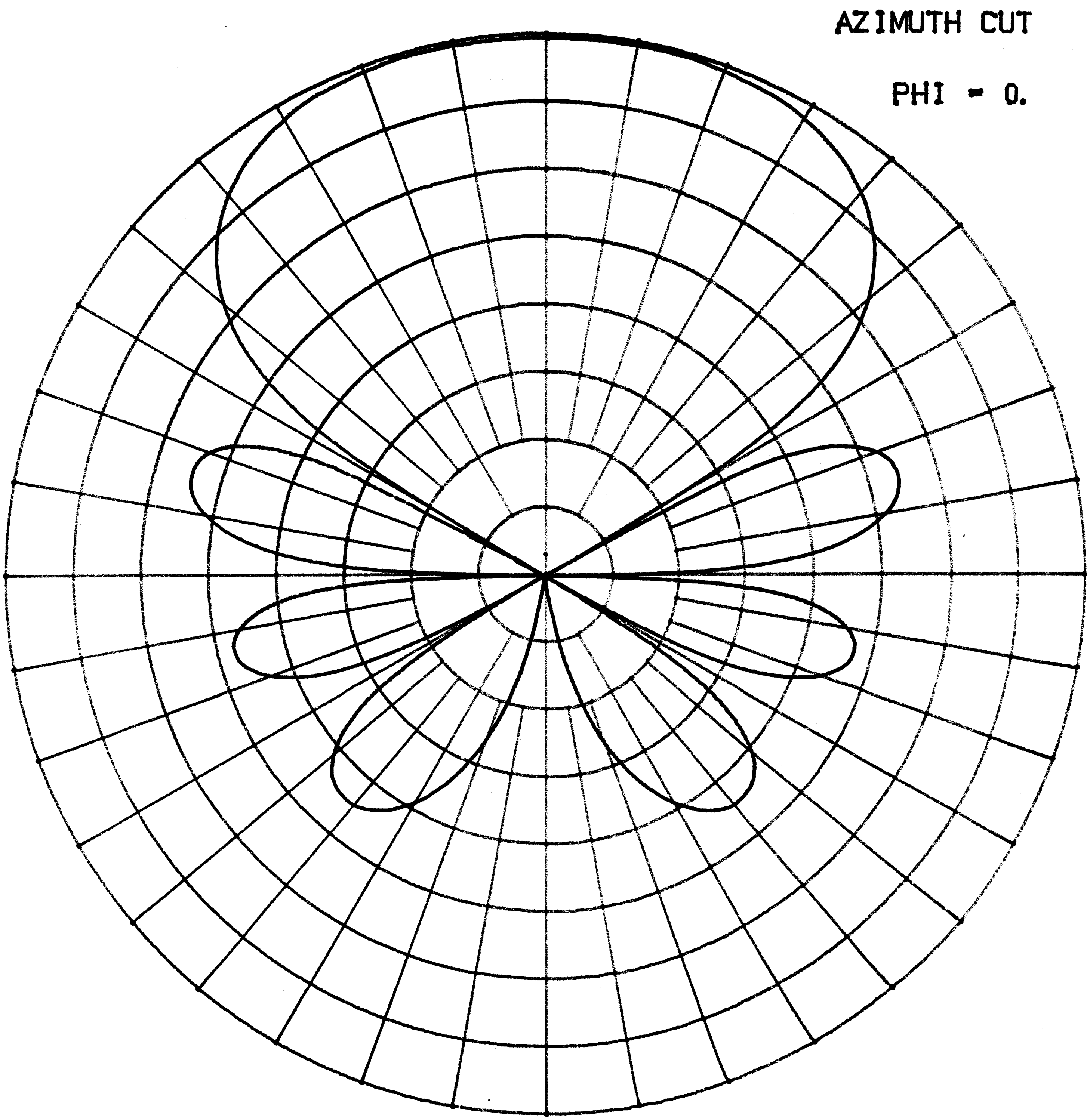
On page 107j is a polar diagram of a scanned array with 10 elements and a spacing of 0.75 wavelengths between elements. The phase shift between elements is 60^o which puts the beam peak at $\theta = 102.8^o$ or 12.8^o off broadside. For the larger spacing between elements it takes larger phase shifts between elements to scan the beam. The same phase shift between elements of a 0.5 wavelength spaced array would scan the beam 19.5^o off broadside. A polar diagram of the same array with an interelement phase shift of 90^o is drawn on page 107k. The beams have been shifted 19.5^o off broadside but another beam has been formed with the same amplitude as the "main" beams at $\theta = 0$. This beam is called a grating lobe. If we continue to scan the beam to further angles off broadside, the dashed curve will shift to the right and the grating lobe will split into two beams and scan away from $\theta = 0$. This means that if we have a phased array with the spacing between elements greater than 0.5 wavelengths, then the beam may be scanned on a limited range before grating lobes occur. The scan angle at which the grating lobe appears is given by the following formula.

$$\theta = \cos^{-1} \frac{360^o - \beta d}{\beta d} \qquad \text{since } \delta = 360^o - \beta d$$

4 ELEMENTS SPACED 0.5 WAVELENGTHS, -90 PHASE SHIFT



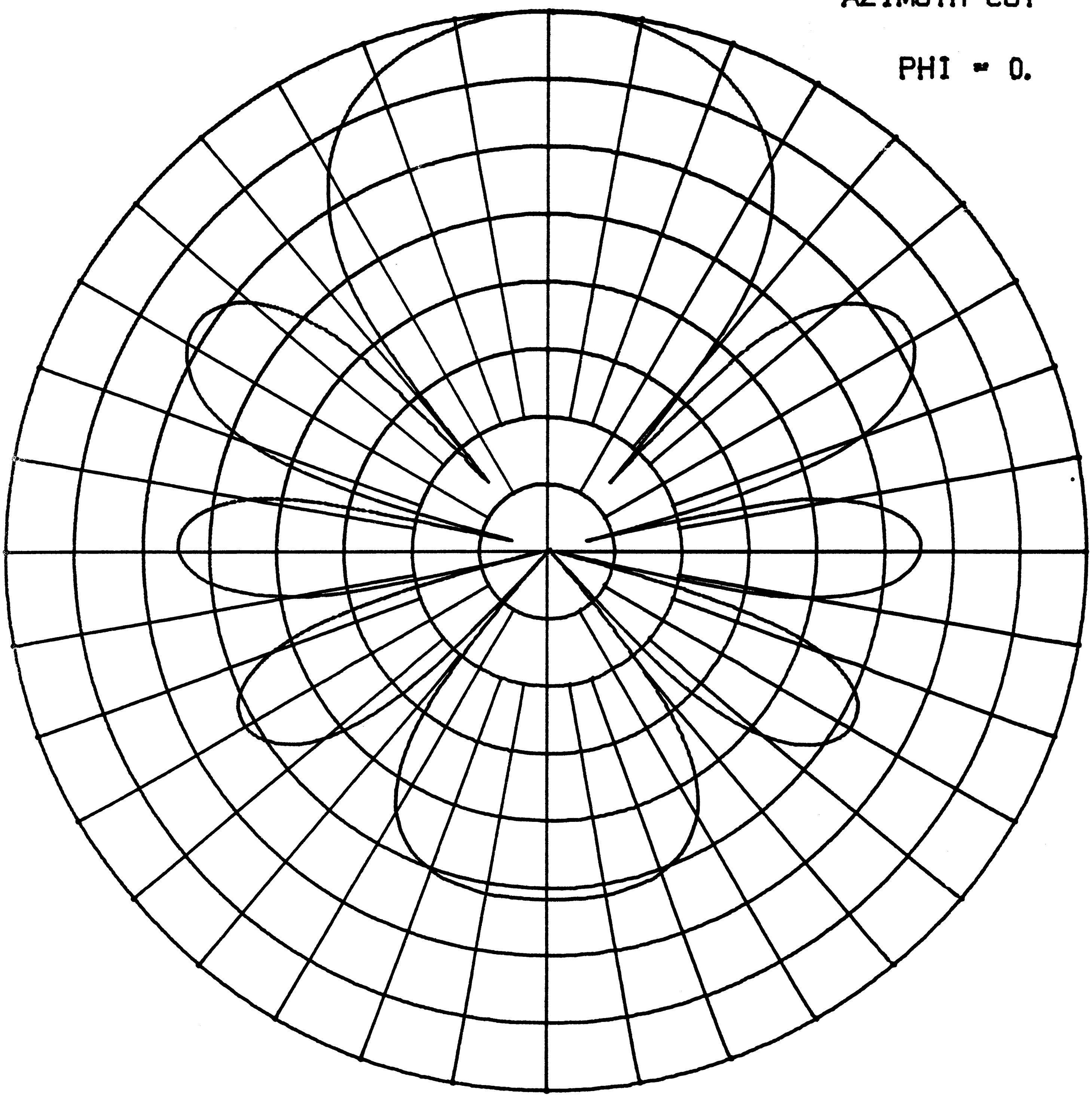
8 ELEMENT ENDFIRE ARRAY



8 ELEMENT INCREASED DIRECTIVITY ENDFIRE ARRAY

AZIMUTH CUT

PHI = 0.



The radiation intensity is the square of this expression.

$$U = E^2 = \left(E_0(\theta,\phi)\right)^2 \left(\sum_i^N E_i e^{j\beta d_i(\theta,\phi)}\right)^2$$

We can recognize the first expression of the product to be the radiation intensity of each element. The second is the radiation intensity function of an array of isotropic antennas. The factor of the pattern of an array of isotropic sources is sometimes called the space factor or array factor. To find the pattern of an array of real antennas, we find the array pattern of an array of isotropic antennas and multiply this by the normalized pattern of the element. Remember the assumption has been made that every element of the array has the same pattern and that each one is pointing in the same direction. In general the array of unidentical antennas can be found from:

$$E = \sum_i^N E_i(\theta,\phi)\, e^{j\beta d_i(\theta,\phi)} \qquad \beta = \frac{2\pi}{\lambda}$$

Where $E_i(\theta,\phi)$ is the voltage pattern, including phase, of the i-th antenna in the direction (θ,ϕ). This is a tedious process, but the computer does not mind.

SPACE ARRAYS

We have only considered arrays with the elements spaced along a single axis (usually Z axis). Antennas can be arrayed in general space arrays. We will look on these as linear arrays of arrays. If we consider one of the axes of the array, we can treat the antenna array along the other axis as the element pattern for the linear array. We can do this from pattern multiplication. If $S_x(\theta,\phi)$ is the space factor for an array along the X axis and $S_y(\theta,\phi)$ is the space factor for an array along the Y axis, then the space factor of the whole array is the product of the two space factors: $S_x(\theta,\phi)\, S_y(\theta,\phi)$. For the general array we have

$$E(\theta,\phi) = E_0(\theta,\phi)\, S_x(\theta,\phi)\, S_y(\theta,\phi)\, S_z(\theta,\phi)$$

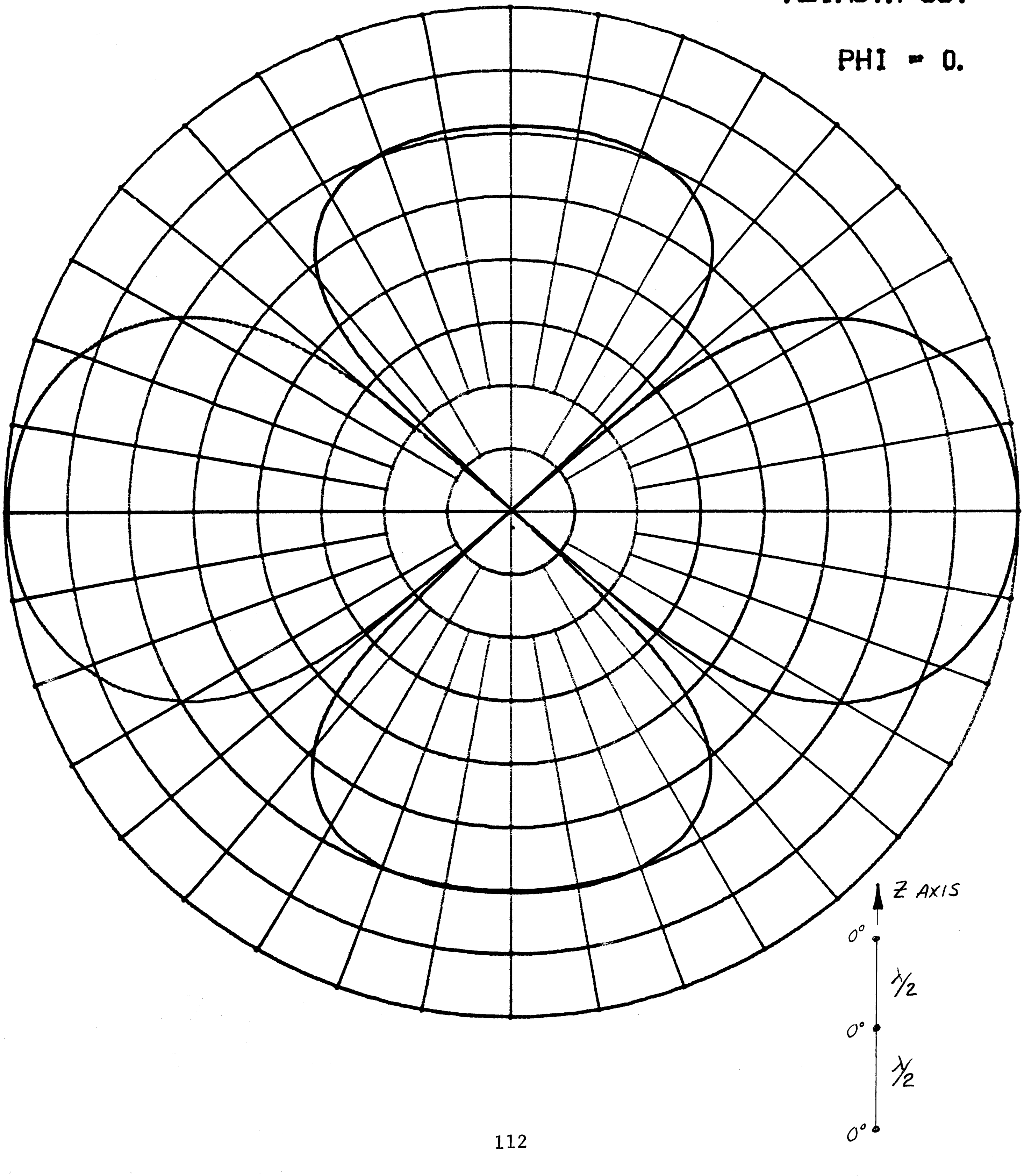
Where $S_x = \sum_i^N E_i e^{j\beta x_i(\theta,\phi)}$, etc.

We can use this pattern multiplication to synthesize space arrays. Example: For the example we will consider a six element array three elements long on the Z axis and two deep on the X axis. The object is to get a single main lobe. Suppose we string out the three antennas at half wavelength spacings along the Z axis and feed them equally. The pattern of such an array is given on page 112. We need to eliminate the lobe on the negative X axis. This can be done by forming an endfire array in the X axis. If we put another row of elements a quarter-wave behind the first set and feed them at 90° out of phase with respect to the first set, then an endfire array will be formed. The pattern of the array along the X axis is given on page 104. When we combine the two array

THREE ELEMENT BROADSIDE ARRAY .5 SPACINGS

AZIMUTH CUT

PHI = 0.



factors, the pattern on page 114 is obtained. The front lobe is pretty much the same but the backlobe has almost been completely eliminated; the little sidelobes are quite insignificant. The two lobes at $\theta = 0$ and $\theta = 180^o$ have been reduced as well. In the other principle plane cut we still have the basic two element quarter-wave spaced endfire array pattern given on page 104.

SIDELOBES AND NONUNIFORM AMPLITUDE ARRAYS

We have only considered arrays which have uniform amplitude excitation. The general patterns of these arrays are given on page 106. For these arrays there is one or more beams of equal amplitude. The lobes below these beams are called sidelobes. Note that the maximum sidelobes tend toward 13 dB below the main beam as the array size grows.

In a radar antenna this level of sidelobes is objectionable because a jamming signal can be transmitted into the sidelobe and confuse the system. Also large reflectors will appear to be at a different place because the system assumes the reflections are in the main beam. Without some adaptive array processing, the sidelobes limit the dynamic range of the radar receiver.

The sidelobes can be reduced by putting an amplitude taper on the array. A distribution based on the binomial coefficients will give a pattern with no sidelobes. The binomial coefficients are found from the equation.

$$(X + Y)^{N-1} = X^{N-1} + (N - 1) X^{N-2}Y + \frac{(N - 1)(N - 2)}{2!} X^{N-3}Y^2 + \dots$$

For N equal six the coefficients are 1, 5, 10, 10, 5, 1. The plots on pages 115 and 116 are the patterns of the 6 element array with and without taper. We have paid a price for the eliminated sidelobes because the beamwidth has increased from 17.2^o to 27^o and the gain has been reduced.

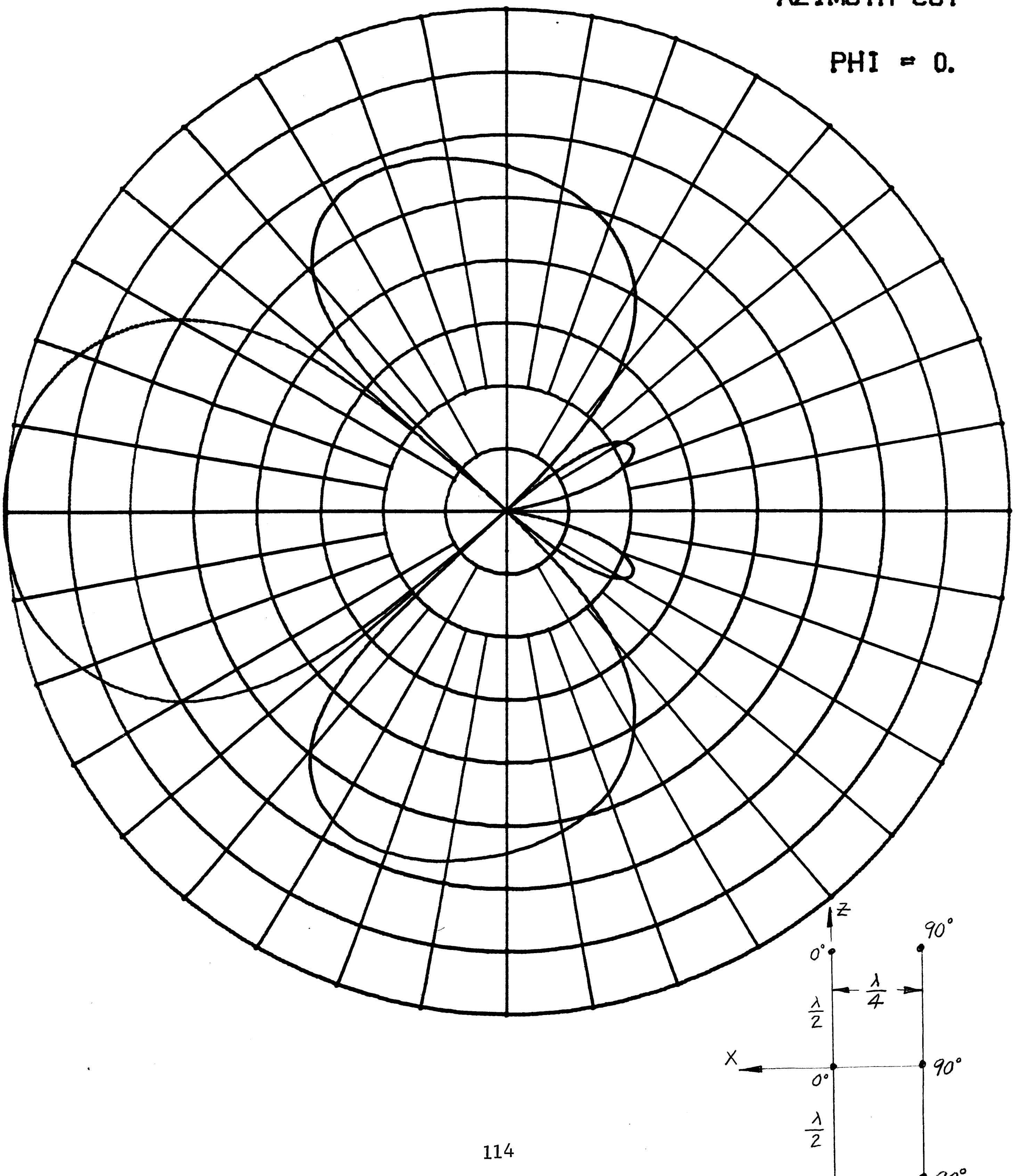
The other problem with the binomial array is the wide variation of the amplitudes. The coefficients are the excitation voltages; the power ratio of the 6 element array is 100. It is difficult to build the power division network with a large variation in the output power. If this was a six by six planar array then the ratio of the outputs would be 40 dB. A radar antenna would require a much larger array than this and a binomial distribution becomes impossible.

There is a class of optimum arrays called the Dolph Tchebyscheff distribution which gives the maximum gain for a given maximum sidelobe level. In general all the sidelobes are the same size. We will discuss these when we cover array synthesis.

6 ELEMENT SPACE ARRAY, 3 BROADSIDE 2 ENDFIRE

AZIMUTH CUT

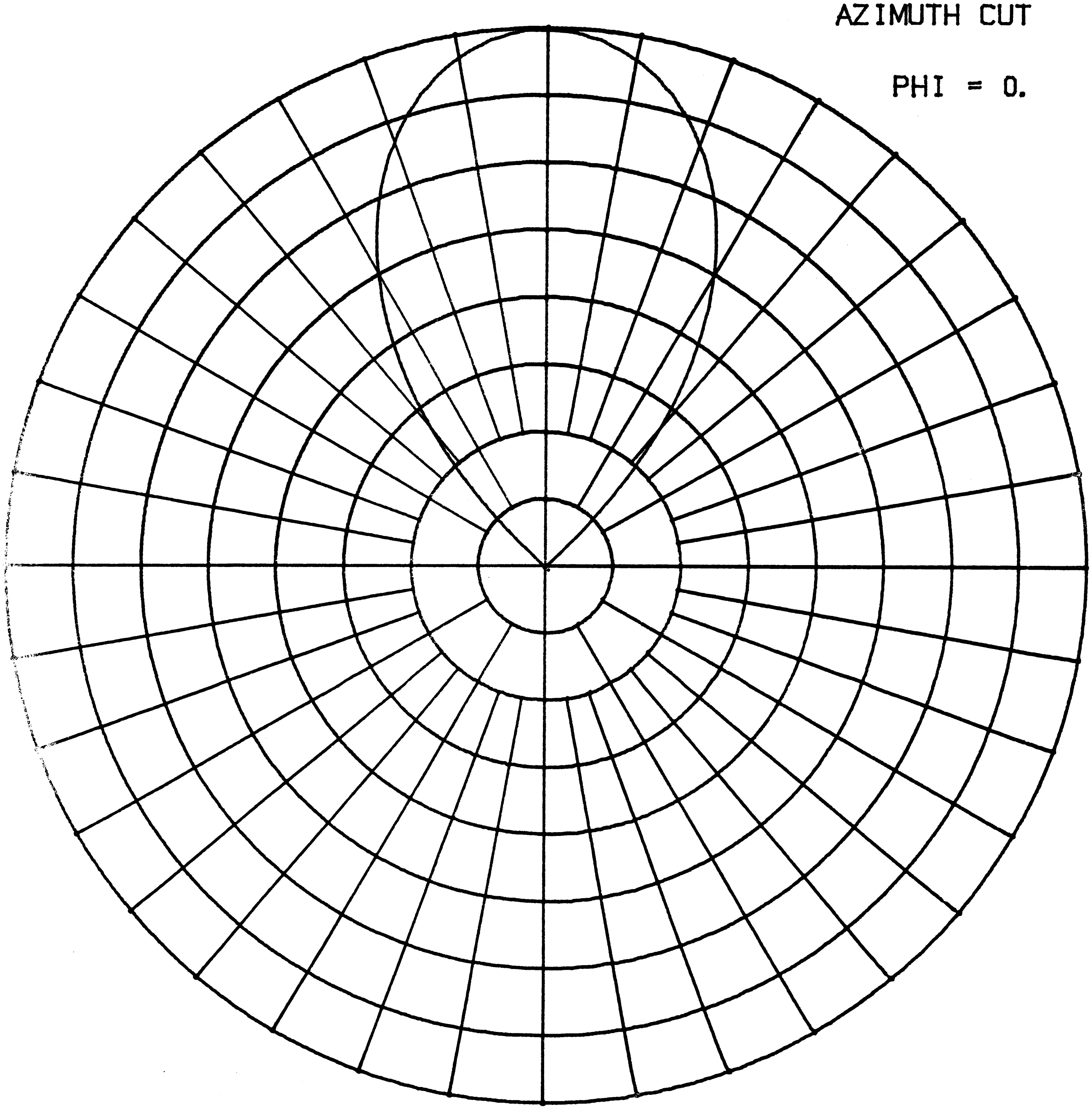
PHI = 0.

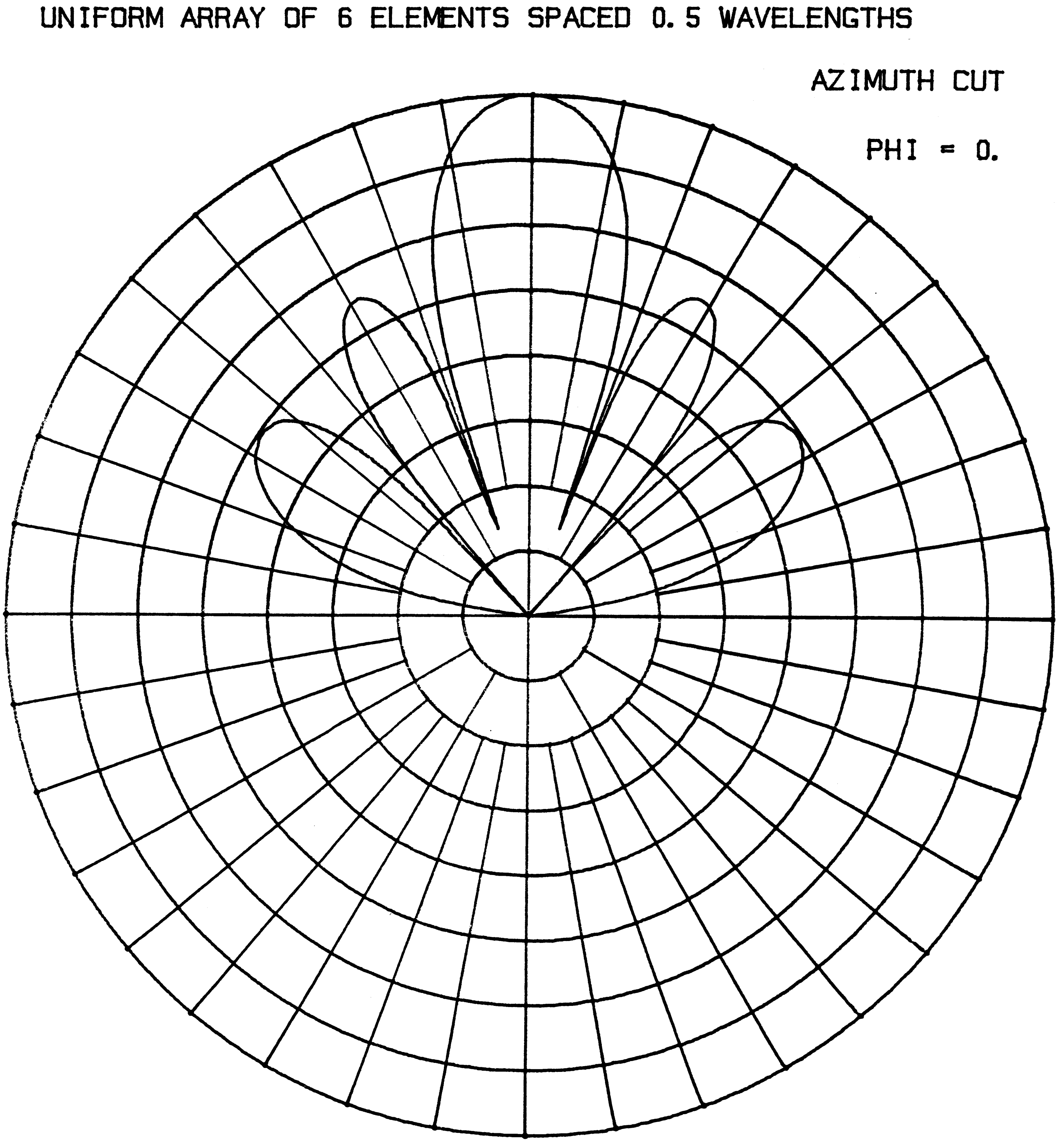


BINOMIAL ARRAY OF 6 ELEMENTS SPACED 0.5 WAVELENGTHS

AZIMUTH CUT

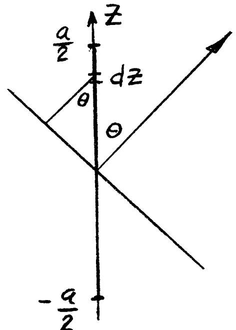
PHI = 0.




UNIFORM ARRAY OF 6 ELEMENTS SPACED 0.5 WAVELENGTHS
AZIMUTH CUT
PHI = 0.

CONTINUOUS ARRAYS

A continuous array is an array of discrete sources where the distance between the elements is infinitesimal and the amplitude is also infinitesimal but there is an infinite number of them. The general formulas for the discrete arrays become integrals.



The figure is a continuous array on the Z axis from $-a/2$ to $a/2$. At some point, Z, the magnitude of the source is $E_o(Z)\,dZ$ on the array. The phase difference from the center of the array is $j\beta Z \cos\theta$. The far field response is given as

$$dE = E_o(Z)\, e^{j\beta Z \cos\theta}\, dZ$$

for this differential source. Notice that the far field approximation has already been applied. To find the response of the array, we must integrate (sum) the differential sources over the limits of the array.

$$E = \int_{-a/2}^{a/2} E_o(z)\, e^{j\beta z \cos\theta}\, dz$$

Where $E_o(Z)$ is the far field response of the electric field for each point source on the array at Z.

UNIFORM CONTINUOUS ARRAY

The easiest problem is the uniform continuous array where $E_o(Z) = E_o$, a constant. The integral becomes

$$E = E_o \int_{-a/2}^{a/2} e^{j\beta z \cos\theta}\, dz = \frac{E_o}{j\beta \cos\theta}\, e^{j\beta z \cos\theta}\Big|_{-a/2}^{a/2}$$

$$E = \frac{E_o}{j\beta\cos\theta}\left[e^{j\beta\frac{a}{2}\cos\theta} - e^{-j\beta\frac{a}{2}\cos\theta}\right]$$

Using the Euler's Identity $\sin b = \frac{1}{j\,2}\left(e^{jb} - e^{-jb}\right)$, the equation reduces to

$$E = \frac{2E_o}{\beta\cos\theta}\sin\left(\beta\frac{a}{2}\cos\theta\right)$$

Let $\psi = \beta a \cos\theta$, then the equation becomes

$$E = E_o a \frac{\sin(\psi/2)}{\psi/2}$$

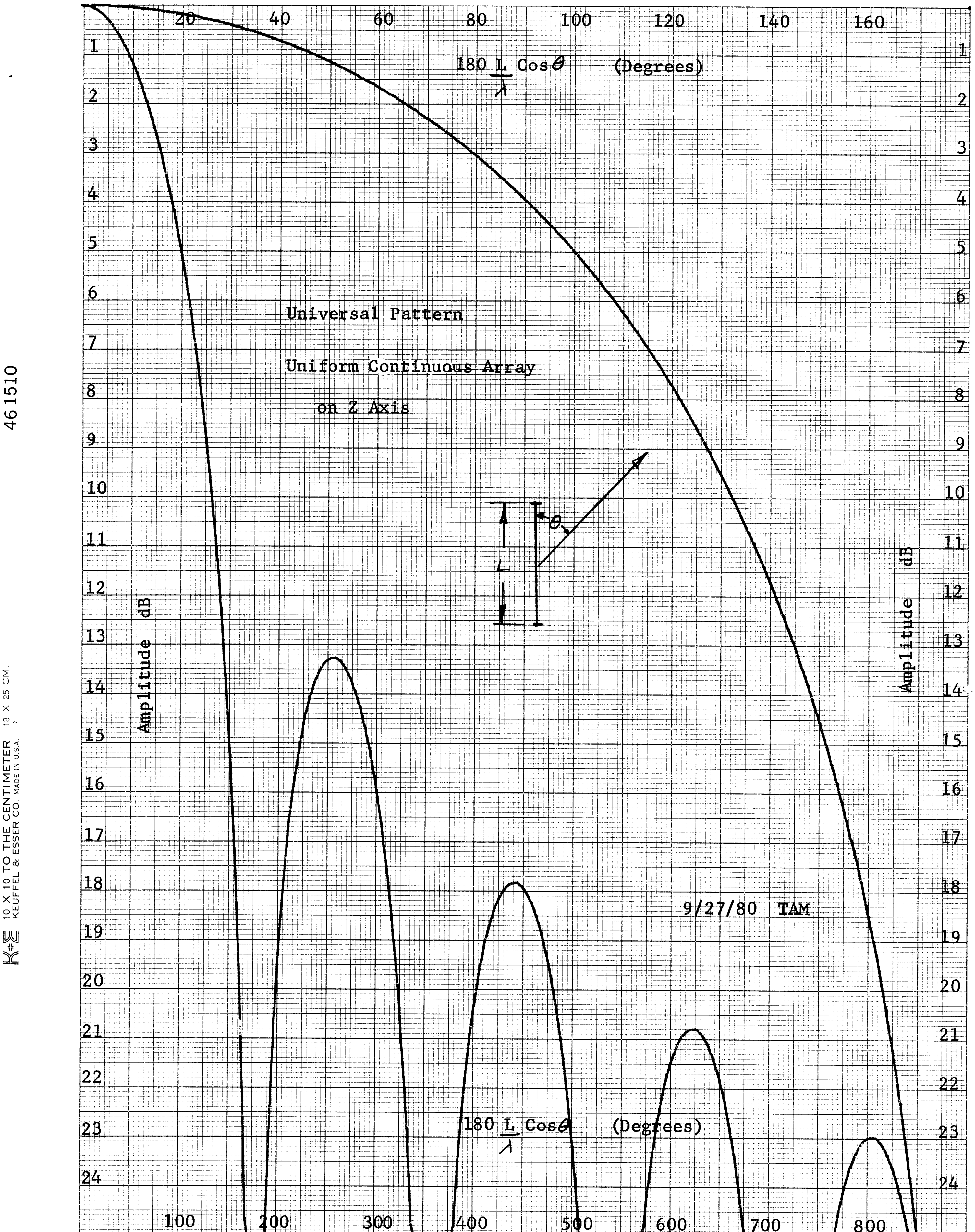
If we normalize this, it becomes $E = \frac{\sin(\psi/2)}{\psi/2}$. A universal pattern of this function is plotted on page 119. We can see that the first sidelobe is down 13.3 dB from the main beam. The peak of the beam is at $\theta = 90^{\circ}$ where all the infinitesimal sources add in phase.

The beamwidth can be found from this curve.

$$\text{Beamwidth} = 2\ \text{Sin}^{-1}\ \frac{79.5\lambda}{180\ \ L} \simeq \frac{159\ \lambda}{L}$$

Where L is the length of the continuous array.

We will discuss this again when we study the radiation from apertures which is a continuous array in two dimensions. For that we will use pattern multiplication of a continuous array of a continuous array to obtain the pattern.


20
40
60
80
100
120
140
160
180 L/λ Cos θ (Degrees)
Universal Pattern
Uniform Continuous Array
on Z Axis
θ
L
Amplitude dB
Amplitude dB
9/27/80 TAM
180 L/λ Cos θ (Degrees)
100
200
300
400
500
600
700
800
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
46 1510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.