MAGNETIC CURRENTS AND CHARGES

From Maxwell's equations we have seen that there are no magnetic charges or currents. Nevertheless it is convenient to introduce them becuase

$$\nabla \cdot \overline{B} = 0$$

they add symmetry to the equations and give solutions from symmetry relationships. The magnetic currents can be used with an electric vector potential analogous to the magnetic vector potential for finding the fields of slots. We denote the magnetic current density $\overline{M}$ which is similiar to the current density $\overline{J}$. The currents through a surface are found from the surface integral of the current densities.

$$i = \iint \overline{J} \cdot d\overline{S} \qquad k = \iint \overline{M} \cdot d\overline{S}$$

GENERALIZED MAXWELL'S EQUATIONS

Adding the magnetic current to Maxwell's equations they become:

Faraday's law: $\nabla \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t} - \overline{M}$

$$\oint \overline{E} \cdot d\overline{\ell} = -\frac{\partial}{\partial t} \iint_S \overline{B} \cdot d\overline{S} - \iint_S \overline{M} \cdot d\overline{S}$$

Ampere's law:

$$\nabla \times \overline{H} = \frac{\partial \overline{D}}{\partial t} + \overline{J}$$

$$\oint \overline{H} \cdot d\overline{\ell} = \iint \overline{J} \cdot d\overline{S} + \frac{\partial}{\partial t} \iint_S \overline{D} \cdot d\overline{S}$$

Gauss's law: $\nabla \cdot \overline{D} = \rho$

$$\iint_S \overline{D} \cdot d\overline{S} = \iiint_V \rho \, dV$$

Magnetic Gauss's law: $\nabla \cdot \overline{B} = \rho_m$

$$\iint_S \overline{B} \cdot d\overline{S} = \iiint \rho_m \, dV$$

Equations of Continuity

$$\nabla \cdot \overline{J} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \overline{M} = -\frac{\partial \rho_m}{\partial t}$$

Above are the generalized Maxwell's equations including magnetic currents, M, and magnetic charges, ρ_m. Using these we can define a vector potential and simplify it until it only gives far field solutions.

ELECTRIC VECTOR POTENTIAL

The two generalized Maxwell's curl equations can be expressed in phasor notation.

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} - \bar{M} \qquad \nabla \times \bar{H} = j\omega\epsilon\bar{E} + \bar{J}$$

Assume we are in a charge free region so that

$$\nabla \cdot \bar{E} = 0$$

We can then express the electric field as the curl of some vector function. The traditional choice is

$$\bar{E} = -\nabla \times \bar{F}$$

Substituting in the second curl equation this becomes

$$\nabla \times \bar{H} = -j\omega\epsilon(\nabla \times \bar{F}) + \bar{J}$$

$$\nabla \times (\bar{H} + j\omega\epsilon\bar{F}) = \bar{J}$$

Because we are in a charge and current free region we can let $\bar{J} = 0$.

A vector whose curl is zero can be derived from the gradient of a scalar function.

$$\bar{H} + j\omega\epsilon\bar{F} = -\nabla\phi_m$$

ϕ_m is called the magnetic scalar potential, the dual of the electric scalar potential. Now substitute $\bar{F}$ in the first curl equation.

$$\nabla \times \nabla \times \bar{F} = j\omega\mu\bar{H} + \bar{M}$$

$$\bar{H} = -\nabla\phi_m - j\omega\epsilon\bar{F}$$

Substituting the equation for the magnetic field, we can eliminate it from the equation of the vector potential, F .

$$\nabla \times \nabla \times \bar{F} - \omega^2\mu\epsilon\bar{F} = \bar{M} - j\omega\mu\nabla\phi_m$$

The term curl curl can be expanded to

$$\nabla \times \nabla \times \bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla^2\bar{F}$$

The equation of the electric vector potential becomes

$$\nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F} - \omega^2\mu\epsilon\bar{F} = \bar{M} - j\omega\mu\,\nabla\phi_m$$

Because of the arbitrariness of F we can pick the divergence of the electric vector potential arbitrarily. We will choose to reduce the equation above by picking:

$$\nabla\cdot\bar{F} = -j\omega\mu\,\phi_m$$

The differential equation then reduces to

$$\nabla^2\bar{F} + \omega^2\mu\epsilon = -\bar{M}$$

We can identify $\omega^2\mu\epsilon$ as β^2. Using this electric vector potential with magnetic currents, the electric and magnetic fields can be found from:

$$\bar{E} = -\nabla\times\bar{F}$$

$$\bar{H} = -j\omega\epsilon\bar{F} + \frac{\nabla(\nabla\cdot\bar{F})}{j\omega\mu}$$

Similiar to the magnetic vector potential, the electric vector potential is found from the magnetic currents.

$$\bar{F} = \iiint \frac{\bar{M}'e^{-j\beta|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}\,dv'$$

This is the dual of the of the magnetic vector potential.

The far field of a magnetic current distribution can be derived from the electric vector potential when the far field radiation approximation is applied to the equation for the potential.

$$\bar{H} = -j\omega\epsilon\bar{F}$$

Where the far field F becomes

$$\bar{F} = \frac{1}{4\pi r}\iiint \bar{M}'e^{-j\beta|\bar{r}-\bar{r}'|}\,dv'$$

The magnetic field is in the same direction as the electric vector potential. The electric field is found from the magnetic field because it will be orthogonal to the direction of propagation and the magnetic field.

$$|E| = \eta|H|$$

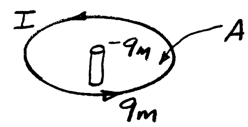
The field from both electric and magnetic currents can be found by considering each current separately with the other zero. In that case the fields from each type of source can be found from the corresponding vector potential. Since the fields are linear, the total fields are the sum of the partial fields from each source.

All this is nice, but we have not seen any magnetic currents. Remember they do not exist. We can approximate certain current distributions as magnetic current elements and distributions of electric fields on a boundary can be replaced by magnetic currents from boundary conditions.

SMALL CURRENT LOOP

The small current loop can be analyzed as a magnetic current element. This loop of current is equivalent to a magnetic dipole.

$$q_m \ell = \mu I A$$



where I is the uniform current in the loop, A is the area of the loop, and q_m the magnetic charge. The magnetic charge is related to the magnetic current by the continuity equation.

$$I_m = -\frac{\partial q_m}{\partial t}$$

Therefore the equivalent magnetic current element is

$$I_m \ell = +\frac{\partial q_m}{\partial t}\ell = j\omega\mu IA$$

The magnetic current density is

$$\bar{M} = I_m \ell\, \delta(\bar{r}')\, \bar{a}_z = j\omega\mu IA\, \delta(\bar{r}')\, \bar{a}_z$$

where $\delta(\bar{r}')$ is the Dirac delta function. Using this the electric vector potential of the magnetic current element is

$$F = \frac{j\omega\mu IA}{4\pi r} e^{-j\beta r}\, \bar{a}_z$$

The fields of a small current loop are found using this electric vector potential and $\bar{a}_z \cdot \bar{a}_\theta = -\sin\theta$.

$$E_\phi = \frac{-j\omega\mu IA}{4\pi} e^{-j\beta r}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)\sin\theta$$

$$H_r = \frac{j\omega\mu IA}{4\pi} e^{-j\beta r}\left(\frac{1}{\eta r^2} + \frac{1}{j\omega\mu r^3}\right)\cos\theta$$

$$H_\theta = \frac{j\omega\mu IA}{4\pi} e^{-j\beta r}\left(\frac{j\omega\epsilon}{r} + \frac{1}{\eta r^2} + \frac{1}{j\omega\mu r^3}\right)\sin\theta$$

These expressions include both the near and far field terms. As with the small incremental dipole, the radiation terms are those with 1/R dependence.

$$E_\phi = \frac{\omega\mu IA\beta}{4\pi r} e^{-j\beta r}\sin\theta \qquad H_\theta = \frac{-\beta^2 IA}{4\pi r} e^{-j\beta r}\sin\theta$$

The following substitution can be made $\omega\mu = \beta\eta = \frac{2\pi}{\lambda}\eta$

The far field equations for the loop antenna becomes

$$E_\phi = \frac{\eta I A \pi}{\lambda^2 r} e^{-j\beta r} \sin\theta \qquad H_\theta = -\frac{\pi I A}{\lambda^2 r} e^{-j\beta r} \sin\theta$$

The power in the radiated field can be found by integrating the Poynting vector over a sphere centered on the magnetic current element.

$$P_r = -\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, E_\phi H_\theta^* \, d\theta \, d\phi$$

$$P_r = \frac{\eta (IA)^2 \pi^2}{\lambda^4} 2\pi \int_0^{\pi} \sin^3\theta \, d\theta$$

$$P_r = \frac{8\eta\pi^3 (IA)^2}{3\lambda^4}$$

The radiation resistance of the antenna can be found from the radiated power.

$$R_r = \frac{P_r}{|I|^2}$$

Therefore the radiation resistance of the small current loop is

$$R_r = \frac{8\eta\pi^3 A^2}{3\lambda^4} = 31149 \, A^2/\lambda^4$$

Since the incremental magnetic dipole has dual fields from the incremental electric dipole, the patterns are the same but with opposite polarizations. Both have directivity of 1.5 or 1.76 dB.

BOUNDARY CONDITIONS

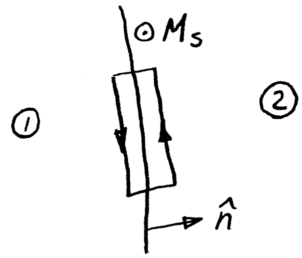
The boundary conditions given on page 152 must be extended if we allow magnetic currents and charges. When we allow a magnetic surface current at the interface between the two mediums, then the tangential electric field is no longer continuous across the boundary. Vectorially this becomes

$$\hat{n} \times (\bar{E}_2 - \bar{E}_1) = -\bar{M}_s$$

where $\bar{M}_s$ is the vector surface magnetic current and $\hat{n}$ is the unit normal vector into medium 2. The tangential electric field discontinuity is given as

$$E_{t2} - E_{t1} = -M_s$$

where M_s is flowing through the surface enclosed by the line integral given in the figure below.



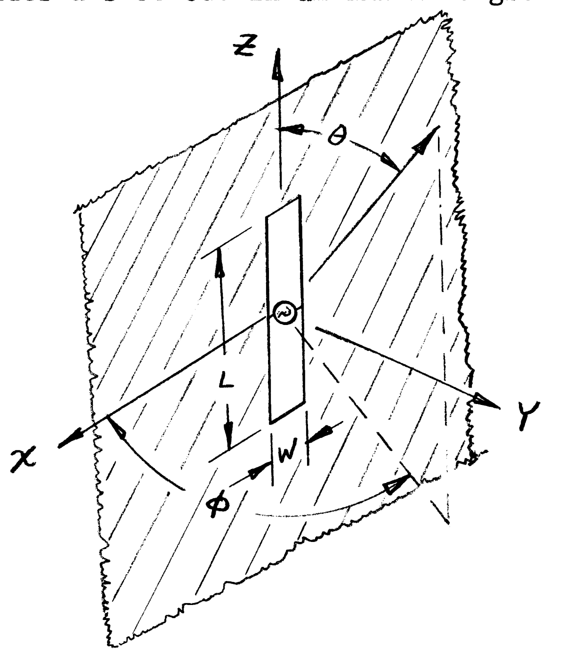
Similiarly when magnetic surface charges are allowed, then the normal component of the magnetic flux density does not need to be continuous across the boundary.

$$\hat{n} \cdot (\bar{B}_2 - \bar{B}_1) = \rho_m$$

The normal component of the magnetic flux density is no longer continuous.

SLOT RADIATOR

Let us consider a slot cut in an infinite ground plane as shown in the figure below.



There is a voltage impressed across the center of the slot along the X axis. The voltage along the slot will be a sinusoidal function; the dual of the dipole. The slot is a transmission line which is shorted at each end. This transmission line is called slot line. The voltage along the line is a standing wave with the voltage given as:

$$V = V_m \sin(\beta(\frac{L}{2} - |z|))$$

Since this is a small width slot, the electric field in the slot can be considered a constant.

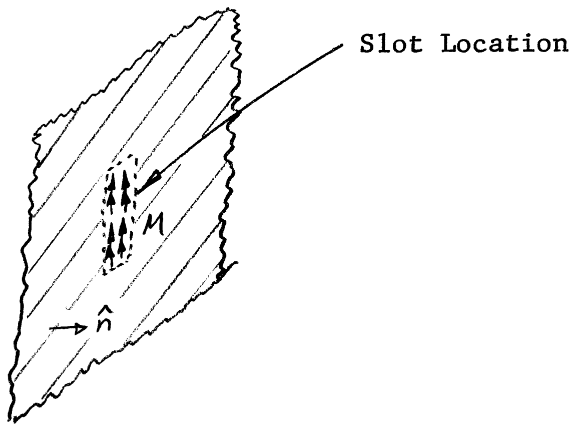
$$E_x = \frac{V_m}{W} \sin\left(\beta\left(\frac{L}{2} - |z|\right)\right)$$

where W is the width of the slot.

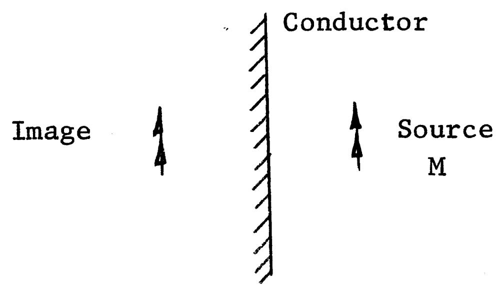
First consider the region $Y > 0$. We can replace the tangential electric field in the slot by an equivalent magnetic current. From boundary conditions we have

$$\overline{M}_s = \overline{E} \times \hat{n}$$

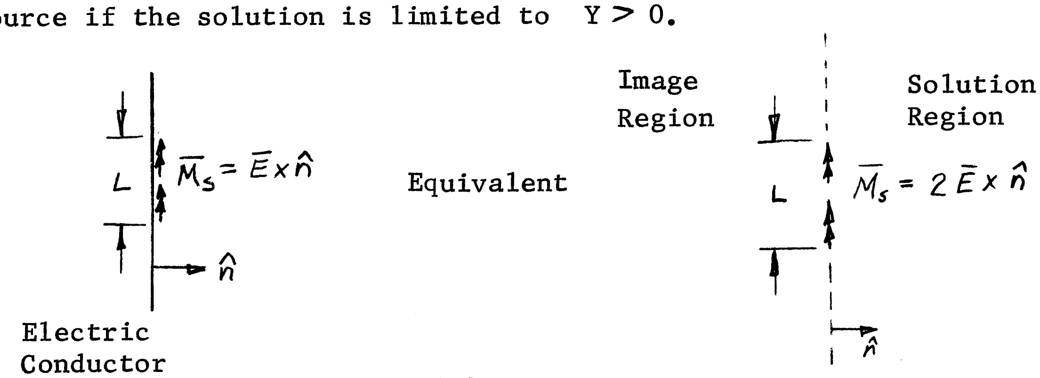
where $\hat{n}$ is the unit vector in the Y direction. This will make the fields zero in the region $Y < 0$, with an electric conductor in place of the slot. We now have a continuous conductor with a magnetic surface current density in the region of the slot.



From image theory for magnetic currents we have the result that the image of a horizontal magnetic current element is also horizontal and in the same direction.



The conductor ground plane behind the slot can be replaced by an equivalent image source if the solution is limited to $Y > 0$.



The magnetic current density is in the Z direction.

$$\overline{M}_s = 2\, E_x\, \overline{a}_z \qquad \text{Since} \quad \overline{a}_z = \overline{a}_x \times \overline{a}_y$$

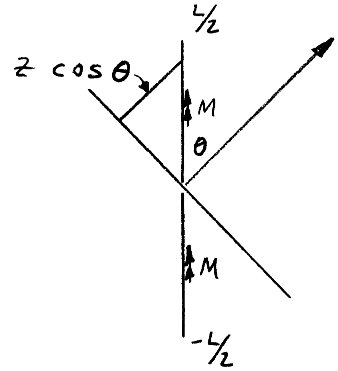
or

$$\overline{M}_s = 2\, \frac{V_m}{W} \,\mathrm{Sin}\left(\beta\left(\frac{L}{2} - |Z|\right)\right) \overline{a}_z$$

We now use the electric vector potential to find the fields.

$$\overline{F} = \frac{e^{-j\beta r}}{4\pi r} \iiint M' e^{-j\beta(\bar{r} - \bar{r}')} dv'$$

Since the slot is small the integral over the X component reduces to just W, the width of the slot.



$$F_z = \frac{e^{-j\beta r} V_m}{2\pi r}\left[\int_0^{L/2} \sin\left(\beta\left(\frac{L}{2} - z\right)\right) e^{j\beta z \cos\theta} dz \right.$$

$$\left. + \int_{-\frac{L}{2}}^{0} \sin\left(\beta\left(\frac{L}{2} + z\right)\right) e^{j\beta z \cos\theta} dz \right] \qquad y > 0$$

The result of the integrals is

$$F_z = \frac{V_m e^{-j\beta r}}{\pi r} \frac{\left[\cos\left(\beta \frac{L}{2}\cos\theta\right) - \cos\left(\beta\frac{L}{2}\right)\right]}{\beta \sin^2\theta}$$

We can find the far field magnetic field from this using $\overline{H} = -j\omega\epsilon\,\overline{F}$

$$H_\theta = \bar{a}_\theta \cdot \bar{a}_z H_z = j\omega\epsilon \sin\theta\, F_z$$

$$H_\theta = \frac{j\omega\epsilon V_m e^{-j\beta r}}{\pi r} \frac{\left[\cos\left(\frac{\beta L}{2}\cos\theta\right) - \cos\left(\frac{\beta L}{2}\right)\right]}{\beta \sin\theta} \qquad y > 0$$

We can reduce this by noting: $\beta = \omega\sqrt{\mu\epsilon}$ and $\eta = \sqrt{\frac{\mu}{\epsilon}}$

$$H_\theta = \frac{j V_m e^{-j\beta r}}{\eta \pi r} \frac{\cos(\beta \frac{L}{2} \cos\theta) - \cos(\beta \frac{L}{2})}{\sin\theta} \qquad y > 0$$

The far field electric field is found from $E_\phi = -\eta H_\theta$

We will now have to find the solution for $Y < 0$. In this region the normal vector to the conductor surface is in the negative Y direction. Using the same arguments about the conductor and image magnetic current densities, the equivalent magnetic current density for $Y < 0$ is

$$\bar{M}_s = -2 E_x \bar{a}_z$$

$$\bar{M}_s = \frac{-2V_m}{W} \sin\left(\beta\left(\frac{L}{2} - |z|\right)\right) \bar{a}_z$$

This is the same magnetic current density as the positive Y region and has the same solution except the sign has been changed.

$$H_\theta = \frac{-j V_m e^{-j\beta r}}{\eta \pi r} \frac{\cos(\beta \frac{L}{2} \cos\theta) - \cos(\frac{\beta L}{2})}{\sin\theta} \qquad y < 0$$

The electric field associated with this magnetic field in the far field is

$$E_\phi = -\eta H_\theta$$

The field is linearly polarized since there is only a E component. This antenna is the dual of the dipole antenna. The radiation intensity is found from

$$U = -E_\phi H_\theta^* r^2$$

$$U = \frac{|V_m|^2}{\eta \pi^2} \left[\frac{\cos(\beta \frac{L}{2} \cos\theta) - \cos(\beta \frac{L}{2})}{\sin\theta}\right]^2$$

The antenna has symmetry about the Z axis so that all great circle cuts are the same and are the same as all the patterns given in the section on radiation from currents. Notice that the phase of the electric field ϕ component changes from 0 to 180 degrees when the plane Y = 0 is closed.

RADIATED POWER AND RADIATION CONDUCTANCE

The total power radiated from the slot is the surface integral of the radiation intensity.

$$P_r = \int_0^{2\pi} \int_0^{\pi} U \sin\theta \, d\theta \, d\phi$$

$$P_r = \frac{|V_m|^2 2\pi}{\eta \pi^2} \int_0^{\pi} \frac{\left[\cos(\frac{\beta L}{2} \cos\theta) - \cos(\frac{\beta L}{2})\right]^2}{\sin\theta} d\theta$$

The radiation conductance of the slot is found from the radiated power.

$$G_R = \frac{P_r}{|V_m|^2}$$

Like the dipole, we have used the maximum response of the sinusoidal voltage as the reference for the radiation conductance.

$$G_R = \frac{2}{\eta\pi}\int_0^{\pi}\frac{\left[\cos\left(\beta\frac{L}{2}\cos\theta\right) - \cos\left(\beta\frac{L}{2}\right)\right]^2}{\sin\theta}\,d\theta$$

This integral is similiar to the radiation resistance of the dipole antenna which is found on page 129.

$$R_{r\,(\text{wire dipole})} = \frac{\eta}{2\pi}\int_0^{\pi}\frac{\left[\cos\left(\beta\frac{L}{2}\cos\theta\right) - \cos\left(\beta\frac{L}{2}\right)\right]^2}{\sin\theta}\,d\theta$$

The two integrals expressions can be equated giving the following result.

$$G_{R\,(\text{slot dipole})} = \frac{4R_{r\,(\text{wire dipole})}}{\eta^2}$$

where η is the characteristic impedance of free space. The radiation resistance of the slot dipole is

$$R_{R\,(\text{slot dipole})} = \frac{\eta^2}{4R_{r\,(\text{wire dipole})}}$$

The input conductance is the conductance seen by the source. The radiation conductance is the ratio of the power radiated to the input voltage.

$$G_i = \frac{P_r}{|V_i|^2}$$

The input voltage is found from the assumption of a sinusoidal voltage wave on the slot. The input voltage is

$$V_i = V_m \sin\frac{\beta L}{2}$$

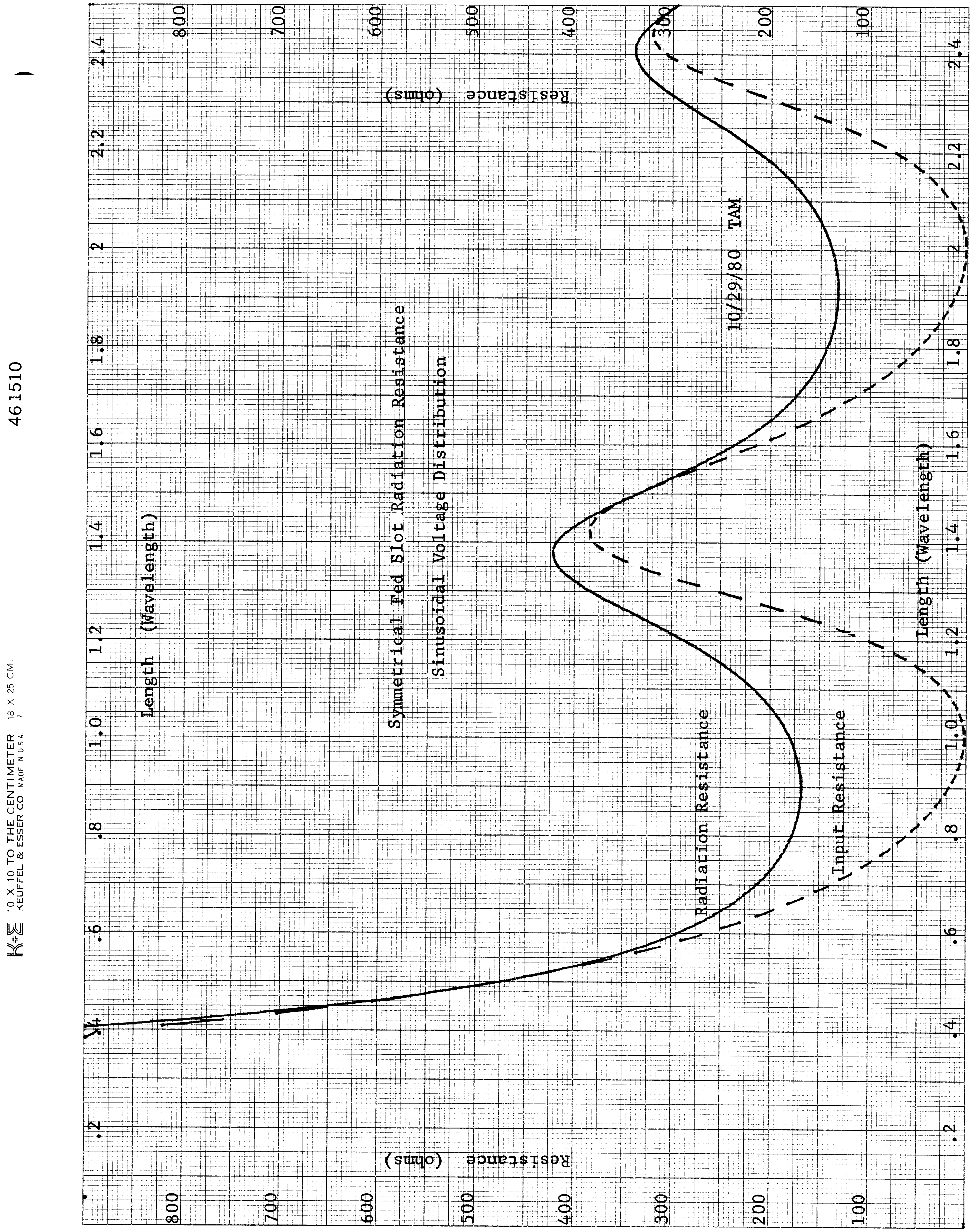
The input conductance can be found for the input voltage and related to the radiation conductance referenced to the maximum sinusoidal voltage wave on the slot.

$$G_i = \frac{G_R}{\sin^2\left(\frac{\beta L}{2}\right)} \qquad R_i = \frac{\eta^2\sin^2\left(\frac{\beta L}{2}\right)}{R_{r\,(\text{wire dipole})}}$$

The radiation resistance referenced to the maximum voltage and the input radiation resistance of the slot dipole are plotted on page 181.

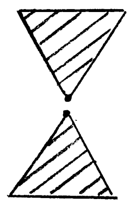
GAIN (DIRECTIVITY)

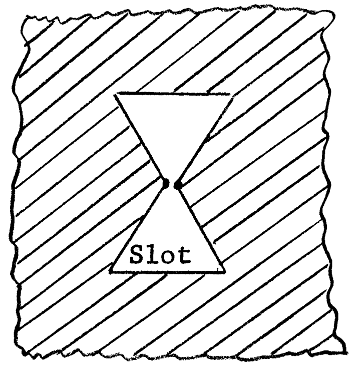
Since the slot dipole has the same pattern as the wire dipole, it will have the same directivity. The curve of directivity is given on page 134.


Symmetrical Fed Slot Radiation Resistance
Sinusoidal Voltage Distribution
10/29/80 TAM
Radiation Resistance
Input Resistance
Length (Wavelength)
Resistance (ohms)
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2
2.2
2.4
100
200
300
400
500
600
700
800
46 1510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.

BABINET'S PRINCIPLE

The dipole and the slot antenna are called complementary structures. The solution of the slot dipole can be found from the equivalent wire dipole and an interchange of the electric and magnetic fields. The following are complementary structures.





Babinet's principle is stated as follows. Suppose the field without a screen is F_o. The field with a screen is F_1 and the field with the complementary screen is F_c. A complementary screen is the interchange of the opaque part and the clear part. Then the following relation holds.

$$F_o = F_1 + F_c$$

The relation seems obvious except it does hold when there is diffraction from the edges of the slot. Booker has extended Babinet's principle to vector electromagnetic fields. Strict complementation of an electrical conductor requires a magnetic conductor which does not exist. Booker has solved this problem by using only perfect conducting infinitesimally thin screens in both cases and an interchange of the electric and magnetic fields between the screen and its complement. If we take two such complementary screens and perform line integrals over the same paths in both cases, we get the result.

$$Z_c = \frac{\eta^2}{4 Z_1}$$

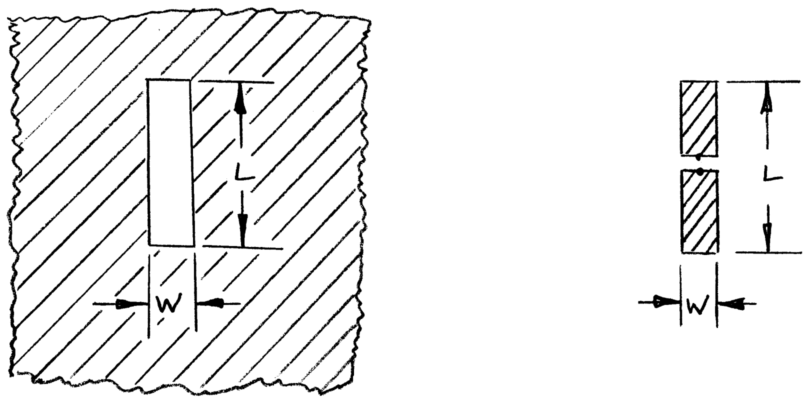
Z_c is the impedance of the complementary structure, Z_1 is the impedance of the structure, and η is the impedance of free space (376.7 ohms). This result includes the whole complex impedance and not just the resistive portion which we saw for the thin dipole and slot complementary structures.

Self Complementary Structure - Many times an antenna will be made from a structure which is self complementary. That is, if the conductors and spaces are interchanged, then the same structure results. The most common case is the flat spiral antenna. The antenna and its complement will have the same input impedance.

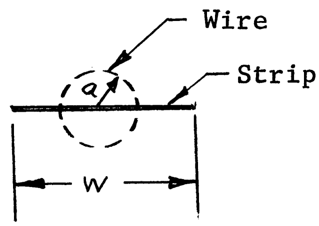
$$Z_0^2 = \frac{\eta^2}{4}$$

This is obtained by rearranging the equation above. When we take the square root, we find the input impedance to be 188 ohms. These antennas are then fed with a 4:1 balun transformer and are fairly closely matched to 50 ohms (1.06 VSWR).

The slot antenna can be made equivalent to the wire dipole and the results for the impedance of a wire dipole can be used for slots. The complement of a slot is a flat strip dipole.



The strip conductor has been related to the round conductor in the following diagram.



$a = W/4$

The equivalent wire has a radius equal to ¼ the stripwidth of the flat conductor. Given a slot radiator, the impedance across the slot is found as follows: An equivalent wire dipole has a diameter one half that of the slot width and the same length. The impedance of the dipole is Z_d. From this the impedance of the slot is

$$Z_s = \frac{\eta^2}{4 Z_d}$$

If we take the resonant thin half-wave dipole antenna, its input impedance is about 67 ohms. The equivalent slot has an input impedance:

$$Z_s = \frac{(376.7)^2}{4\,(67)} = 530 \text{ ohms}$$

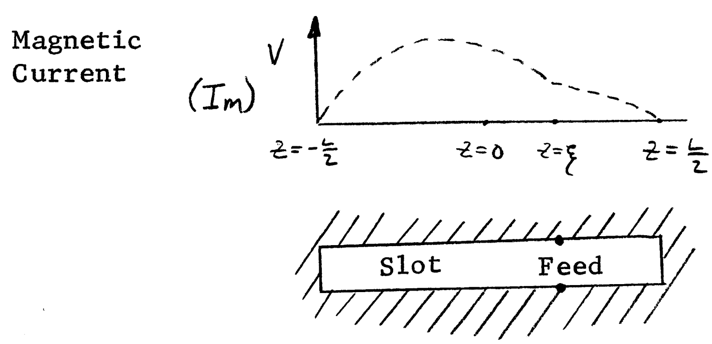
The full half-wave dipole has an input impedance 73 + j 42.5 ohms. The full half-wave slot will have an input impedance of

$$Z_s = \frac{(376.7)^2}{4(73 + j\,42.5)} = 363 - j\,211 \text{ ohms}$$

When the half-wave slot is too long for resonance, its impedance is capacitive; whereas the half-wave dipole is inductive when it is too long for resonance. These are complementary impedances.

The impedance of the slot is so high that it is difficult to feed it from a coaxial line. There are two solutions. In the first solution the length of the slot is increased. The longer equivalent dipole has higher and higher input impedances as the length approaches one-half wavelength; the complementary slot will therefore have lower and lower input impedance as the length is increased toward one-half wavelength. Kraus in Antennas gives an input impedance of 50 ohms for a slot 0.925 wavelengths long and 0.066 wavelengths wide. The equivalent dipole has a length of 0.925 wavelengths and a diameter of ½ of the width of the slot or 0.033 wavelengths.

The second solution is to use an offset feed. On page 136 offset feeds for a dipole was discussed. The results given there hold for the slot in a complementary sense. The same coordinate system as the center fed slot is used for the offset feed.



The voltages at the ends of the slot are zero and increase sinusoidally from the ends. In general there is a discontinuity in the slope of the voltage curve at the feed point but the voltage on each segment must be the same. The voltage across the slot is proportional to the electric field across the slot. The equivalent magnetic current in the slot is proportional to the elctric field. Therefore the equivalent magnetic current is.

$$I_m(Z) = I_{m1} \operatorname{Sin} \beta\left(\frac{L}{2} - Z\right) \qquad Z \geq \xi$$

$$I_m(Z) = I_{m2} \operatorname{Sin} \beta\left(\frac{L}{2} + Z\right) \qquad Z \leq \xi$$

The continuity of the voltage and magnetic current at $Z = \xi$ gives

$$I_{m1} \operatorname{Sin} \beta\left(\frac{L}{2} - \xi\right) = I_{m2} \operatorname{Sin} \beta\left(\frac{L}{2} + \xi\right)$$

This will be satisfied if

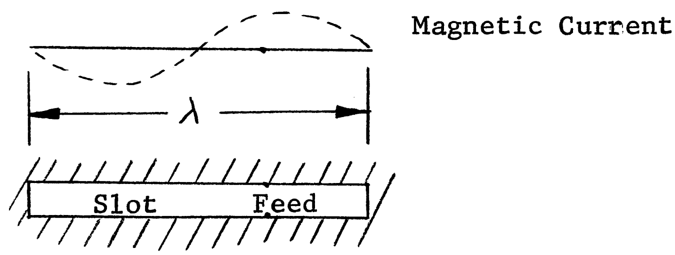
$$I_{m1} = A \operatorname{Sin} \beta\left(\frac{L}{2} + \xi\right) \text{ and } I_{m2} = A \operatorname{Sin} \beta\left(\frac{L}{2} - \xi\right)$$

The result is that the magnetic current in the slot is given by

$$I_m(Z) = A \sin\beta\left(\frac{L}{2} + \xi\right) \sin\beta\left(\frac{L}{2} - Z\right) \qquad Z \geq \xi$$

$$I_m(Z) = A \sin\beta\left(\frac{L}{2} - \xi\right) \sin\beta\left(\frac{L}{2} + Z\right) \qquad Z \leq \xi$$

The fields can be found using this magnetic current and the electric vector potential. We can have the same problem with the one wavelength slot as with the full wavelength dipole. If it is offset fed, then the distribution will be



and there will be a null in the pattern perpendicular to the slot. The easiest way to find the input impedance of the offset fed slot is to use a wire analysis program and take the complementary impedance. When the slot is a resonant half-wavelength long, then the impedance will decrease approximately as

$$\sin^2\beta\left(\frac{L}{2} - \xi\right)$$

as the feed is offset from the center by ξ. Take the resonant half-wave slot with an input impedance of about 510 ohms.

$$L = .45\lambda$$

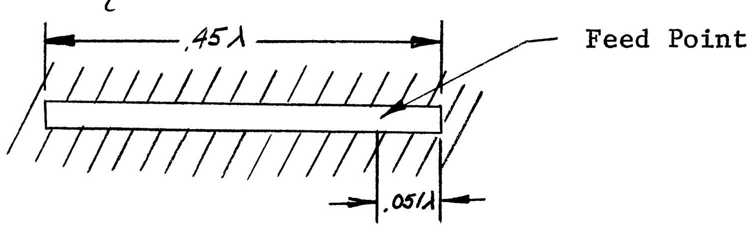
We can solve the above equation for the 50 ohms feed point.

$$\sin^2 \frac{2\pi}{\lambda}\left(\frac{.45\lambda}{2} - \xi\right) = \frac{50}{510}$$

Solving for ξ we get:

$$\xi = \frac{.45\lambda}{2} - \frac{\lambda}{2\pi} \sin^{-1}\sqrt{\frac{50}{510}}$$

$$\xi = .174\lambda$$



SLOTS WHICH RADIATE ON ONE SIDE OF THE GROUND PLANE

All the cases covered above are for slots which radiate on both sides of the ground plane. Most slots are backed by some cavity or transmission line and radiate only on one side of the ground plane. The results given above can be modified to account for this. Just like the monopole, the antenna will only radiate on one side. The radiated power is found for a slot in the X-Z plane by the integral:

$$P_r = \int_0^{\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi$$

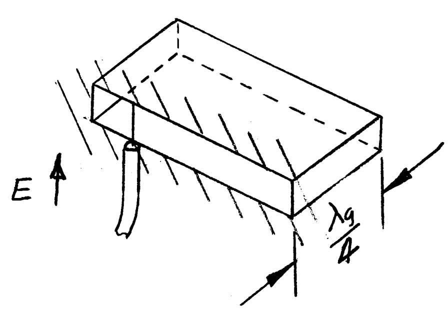
This is one half the power radiated on both sides. The radiation conductance of the slot is found from the radiated power.

$$G_R = \frac{P_r}{|V_m|^2}$$

The maximum sinusoidal voltage across the slot is the same as the slot which radiates on both sides. A slot which radiates on only one side of the ground plane has ½ the radiation conductance of the slot which radiates on both sides. Since the radiation conductance decreased by ½, the resistance is increased by two.

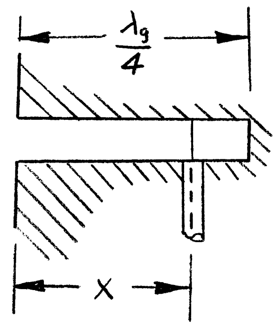
The directivity is four Pi times the maximum radiation intensity divided by the power radiated. Since the power radiated is only one half with the same maximum radiation intensity, the directivity is twice that for the slot which radiates on both sides. The directivity is found from the graph on page 134 by adding 3 dB.

The slot can be limited to radiation on one side of the ground plane if the other side is enclosed in a cavity. For a slot with shorts on the ends of the slot transmission line, a waveguide section can be used for the cavity.



The depth of the slot is a quarter-wave long in the waveguide. We will discuss waveguide relations when horns are covered. The guide can be made longer or shorter to resonate out any shunt admittance of the slot at the desired frequency if need be. If the shunt admittance of the slot is zero, then the quarter-wave shunt shorted stub of the waveguide section will present an open circuit across the input and can be ignored.

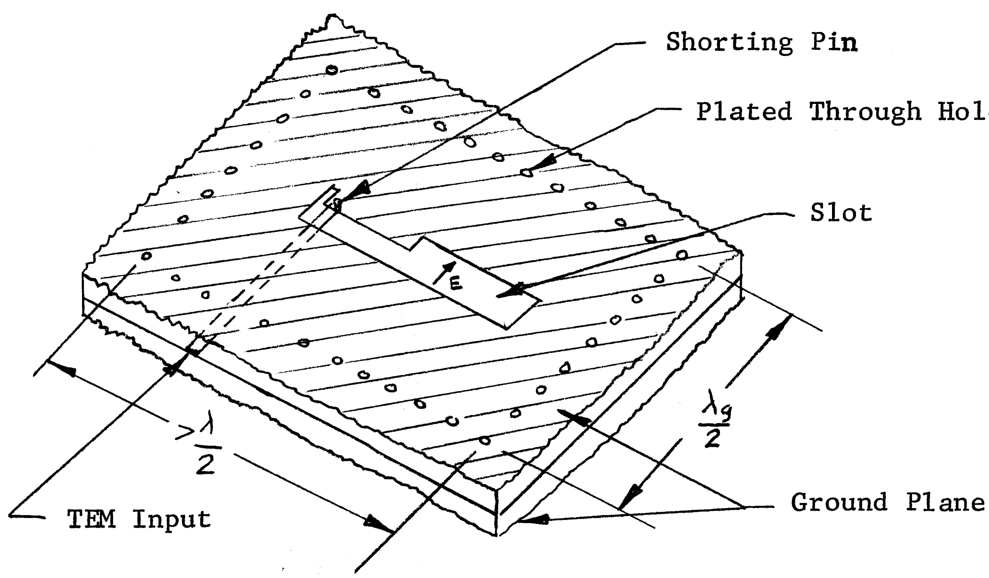
The input impedance of a slot is very high compared with the coax characteristic impedance. If the feed is moved towards the short in the cavity, then the input impedance will be reduced.



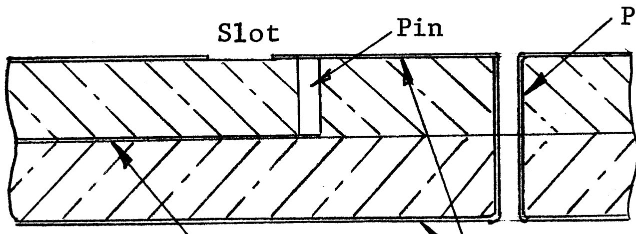
$$R_i \cong R_s \cos\left(\frac{4\pi X}{\lambda}\right)$$

The impedance decreases approximately as the formula given above. The impedance can be decreased further by feeding close to one of the side walls and close to the back wall.

A cavity can also be formed in a stripline structure where the depth behind the slot is small.



The antenna is fed by a stripline circuit from below which can contain a power division network for arrays. Plated through holes are fabricated between the ground planes around the radiating slot which is etched in the top ground plane. These holes form a waveguide cavity between the ground planes. The cavity is fed by a TEM wave between the center conductor and the two ground planes. The center conductor passes under the slot and then a pin is used to short it to the top ground plane. This pin excites the slot by impressing a voltage across it, which is one way of looking at this. The other way is that the loop between the center conductor and the upper ground plane is a magnetic loop which excites the waveguide cavity formed by plated through holes. This method says that the rows of pins must be



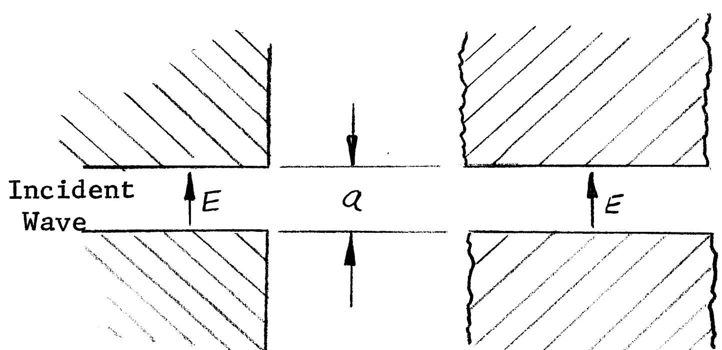
Cross Sectional View of Stripline Fed Slot

spaced greater than one-half wavelength in the dielectric constant to form a resonant cavity. The resonant cavity loses energy from the slot and loads the TEM transmission line.

The bandwidth of this slot is proportional to the thickness of the cavity and the width of the slot. It is generally a narrowband antenna. The bandwidth is mostly related to the thickness of the stripline ground planes and not the width of the slot. The slot must be offset fed to achieve a low impedance from the slot and match it to the transmission line.

PARALLEL PLATE SLOT

Another method of restricting the radiation to one side of the ground plane is to use an aperture in a ground plane fed from a parallel plate waveguide. The parallel plate guide can support a TEM wave.

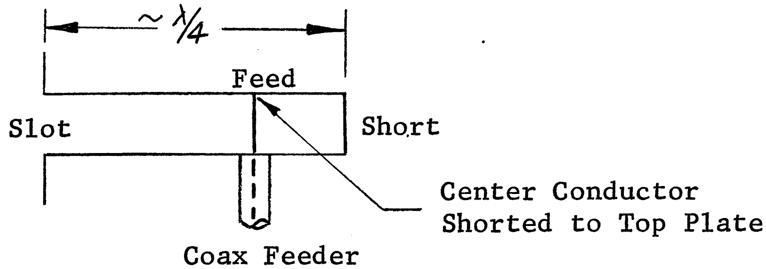


The slot has a uniform field across it without zero fields at the edges. Harrington Time Harmonic Electromagnetic Fields p. 183 has the approximate conductance and susceptance per unit length of slot.

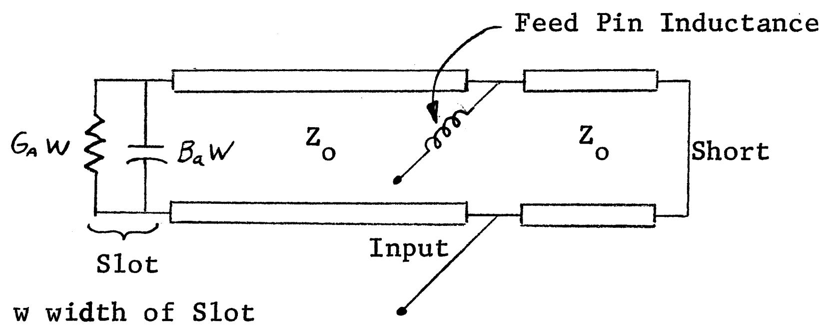
$$G_A = \frac{\pi}{\eta\lambda}\left[1 - \frac{(\beta a)^2}{24}\right] \qquad \frac{a}{\lambda} < 0.1$$

$$B_a = \frac{(3.135 - 2\,Ln(\beta a))}{\lambda\eta}$$

The effect of the susceptance is to make the parallel plate transmission line appear longer at the slot to the effective open circuit. The parallel slot feed looks like the figure below.



This antenna can be analyzed as a tranmission line network:



The impedance of the parallel plate transmission line can be found from the parallel capacitance and the velocity of light in the medium between the plates. From transmission line relations we have

$$Z_0 = \frac{1}{v C}$$

C is the capacitance per unit length and v is the velocity of light.

$$v = \frac{1}{\sqrt{\mu \epsilon}}$$

$$C = \frac{\epsilon W}{a}$$

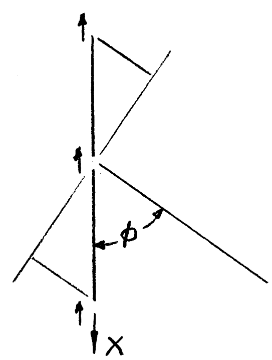
$$Z_0 = \frac{a\sqrt{\mu \epsilon}}{W \epsilon} = \frac{\eta a}{\epsilon_r W} \quad \text{Since } \eta = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

The impedance of the parallel plate guide is usually very small. The feed point can be found to match the coax feed to the slot and parallel plate waveguide structure. It is found that the resonant frequency is a function of the position of the feed. The input impedance is lowered as the feed is moved toward the back short. It usually takes some trial and error process to get the antenna to be matched and resonant at the center frequency. The size of the slot is limited to about 1.9λ in width where the slot impedance is about 50 ohms and the feed is at the edge of the slot. The input impedance follows the expression for impedance given on page 187 for a waveguide fed slot.

EFFECTS OF FINITE GROUND PLANE

It is obvious that the ground plane cannot be infinite. We must deal with the effects of limiting the ground plane. Kraus has stated that the input impedance of a slot antenna will be about the same as with an infinite ground plane if the ground plane extends for at least one wavelength on each side of the slot. The impedance is a second order effect. The pattern will show marked effects from reducing the ground plane.

There are nulls in the pattern of a slot in the directions of the axis of the slot; orthogonal to the electric field. For any reasonable ground plane $(>\lambda/2)$, the H plane pattern will not show the effects of a finite ground plane. The E plane is another matter. With a finite sheet the pattern shows scalloping. As the size of the sheet is increased the number of undulations increases but decreases in amplitude. Since the slot radiates in the directions of the edges in the E plane, by diffraction theory there will be equivalent sources at the edges of the ground plane. The size of these sources is $k\, e^{-j\delta}$ where k is less than one and δ is a phase shift relative to the slot phase. As the size of the ground plane increases the factor k decreases.



The edge sources form a two element array whose response is

$$2K \cos\left(\beta \frac{L}{2} \cos\theta\right)$$

This factor will add to or substract from the slot radiation depending on the phase term at $\theta = 90^{\circ}$. Once this is determined, then the undulations in the pattern are fixed in the coordinate θ. We can determine these by finding the voltage maximums and minimums of the two element array. We must note that there is an axis of symmetry about the slot. The undulations below the axis of the slot will be the same as above. To take advantage of this symmetry we will rotate the coordinate so that $\theta = 0$ occurs on this line of symmetry (former $\theta = 90^{\circ}$). The two element array factor becomes

$$2k \cos\left(\frac{\pi}{\lambda} L \sin\phi\right)$$

This factor will add to the radiation of the slot when the argument of the cosine is $2n\pi$, assuming Cos δ is positive.

$$\frac{\pi}{\lambda} L \sin\phi = 2n\pi$$

The two element array will subtract from the radiation of the slot if the argument of the cosine is $(2n - 1)\pi$, when Cos δ is positive.

$$\frac{\pi}{\lambda} L \sin\phi = (2n-1)\pi$$

$$\text{Max/(Min)} = \sin^{-1}\left(\frac{2n\lambda}{L}\right)$$

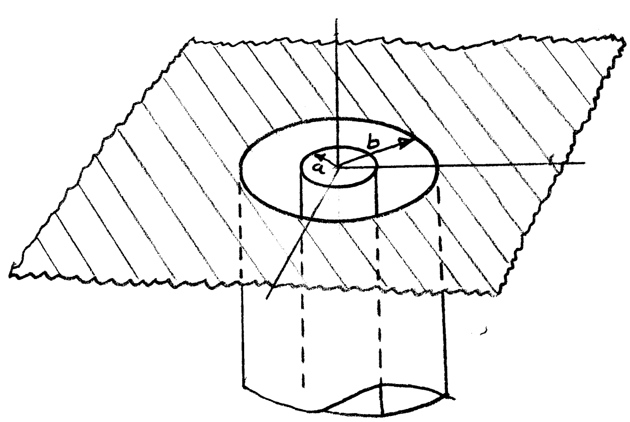
$$\text{Min/(Max)} = \sin^{-1}\left(\frac{(2n-1)\lambda}{L}\right)$$

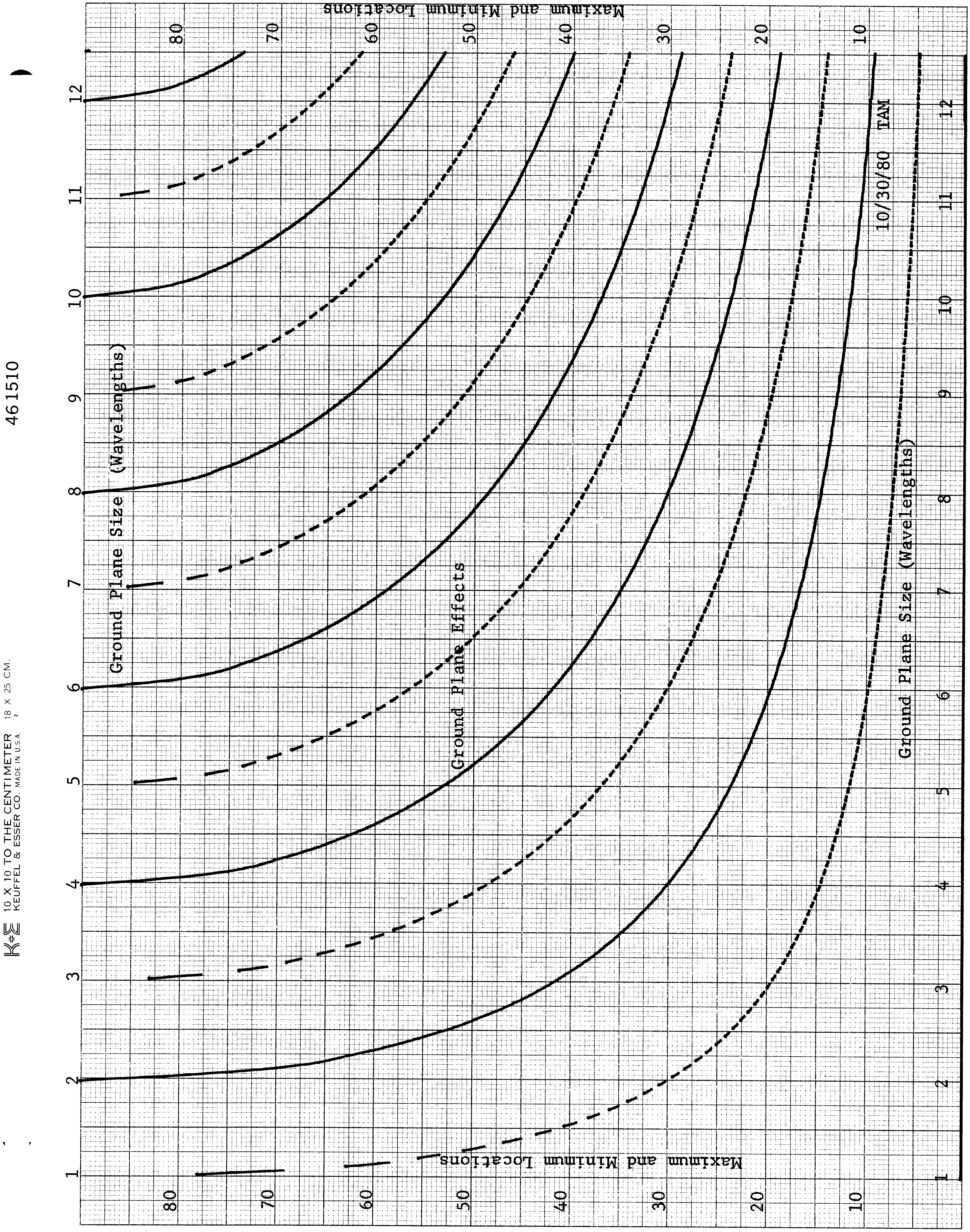
Whether these locations are maximums or minimums depends on whether Cos δ is positive or negative. The positive values are given above; the parenthetical expressions are for negative Cos δ.

A plot for various ground plane sizes is given on page 192. These are locations of maximums or minimums for various sized ground planes. Once the sign of the undulations is determined at boresight, then all the maximums and minimums can be determined from the graph. An example of these undulations is given on the measured plot on page 193 of an E plane pattern on about 3.5λ ground plane. The scalloping is approximately where it is predicted to be from the graph on page 192. This case was for two slots spaced 0.32λ and does not exactly follow the graph.

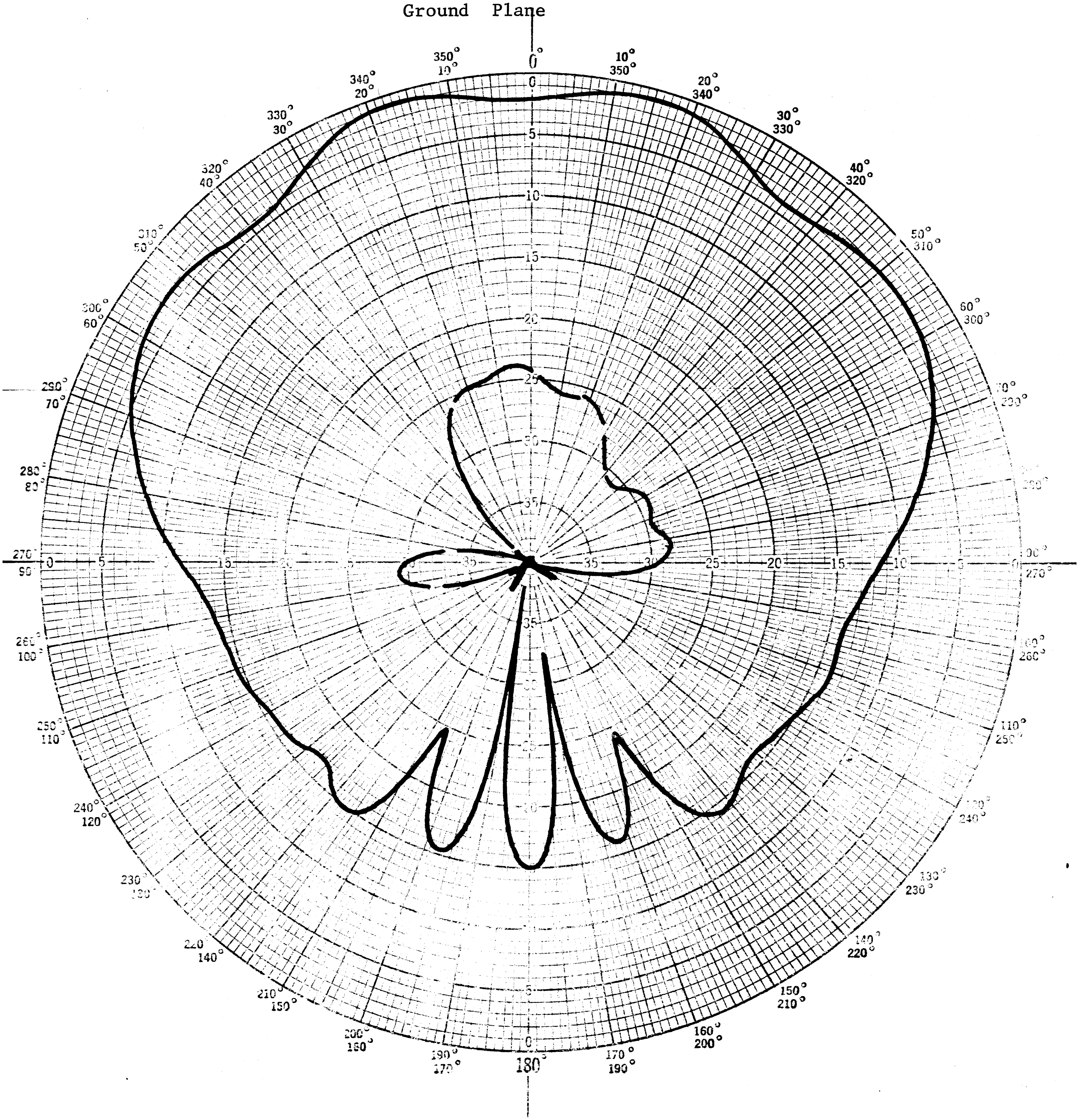
RADIATION FROM COAX

Consider a coax opening into a ground plane. This open circuited coax will radiate some of the energy.




Maximum and Minimum Locations
Ground Plane Size (Wavelengths)
Ground Plane Effects
Ground Plane Size (Wavelengths)
10/30/80 TAM
Maximum and Minimum Locations
46 1510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.

E Plane Pattern of Two Slots Spaced 0.32 Wavelengths on a 3.5 wavelengths Ground Plane



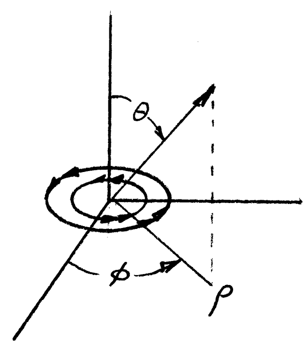
Polar Chart No. 127D
SCIENTIFIC-ATLANTA, INC.
ATLANTA, GEORGIA

The electric field in the open circuited coax is

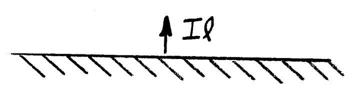
$$E_r = \frac{-V}{\rho' \ln(b/a)}$$

where ρ' is the radius of the coax and V is the voltage across the coax. It is assumed that the field over the aperture is the same as the transmission line mode of the coax. This field can be replaced by a magnetic surface current with zero fields below the ground plane.

$$M_\phi = \frac{V}{\rho \ln(b/a)}$$



This loop of magnetic current density can be analyzed as an equivalent electric dipole over a ground plane.



The total magnetic current moment is

$$KA = \int_0^{2\pi}\int_a^b \frac{\rho^2}{2} M_\phi \, d\rho \, d\phi$$

$$= \frac{\pi V(b^2 - a^2)}{2 \ln(b/a)}$$

The magnetic current moment is equivalent to an electric current element.

$$I\ell = -j\omega\epsilon K A$$

All this has assumed that $b \ll \lambda$. We can find the radiated field by using the method of images and the magnetic vector potential. The source is

$$2 I\ell \, \delta(\bar{r}') = \frac{-j\omega\epsilon\pi V(b^2 - a^2)}{\ln(b/a)} \delta(\bar{r}')$$

where the current element is in the Z direction and $\delta(\bar{r}')$ is the Dirac delta function. The magnetic vector potential is

$$A_z = \frac{-j\omega\epsilon\pi V(b^2-a^2)}{4\pi r\, \ln(b/a)} e^{-j\beta r}$$

The electric field is given by $E_\theta = j\omega\mu A_z \sin\theta$. From page 123 we find the electric field as

$$E_\theta = \frac{\omega\epsilon\eta\pi V(b^2-a^2)}{2 \ln(b/a)\lambda r} e^{-j\beta r} \sin\theta$$

The Poynting vector is found from $|E_\theta|^2/\eta$

$$S_r = \eta \left| \frac{\omega\epsilon\pi V(b^2-a^2)}{2 \ln(b/a)\lambda} \right|^2 \frac{1}{r^2} \sin^2\theta$$

The total power radiated is

$$P_r = \int_0^{2\pi}\int_0^{\pi/2} S_r r^2 \sin\theta\, d\theta\, d\phi$$

The integral over θ is only from 0 to $\pi/2$ since the field is zero below the ground plane.

$$P_r = 2\pi\eta \left| \frac{\omega\epsilon\pi V(b^2-a^2)}{2\lambda \ln(b/a)} \right|^2 \frac{2}{3}$$

since $\int_0^{\pi/2} \sin^3\theta\, d\theta = \frac{2}{3}$ We can recognize $\omega\epsilon = \frac{2\pi}{\lambda\eta}$

which reduces the expression to

$$P_r = \frac{4\pi}{3\eta} \left| \frac{\pi^2(b^2-a^2)V}{\lambda^2 \ln(b/a)} \right|^2$$

The radiation conductance is found from

$$G_R = \frac{P_r}{|V|^2} = \frac{4\pi^5}{3\eta} \left[\frac{b^2-a^2}{\lambda^2 \ln(b/a)} \right]^2$$

The formula is only an approximation because it was assumed that the coax opening is small with respect to a wavelength. The radiation decreases with the fourth power of the wavelength. As the coax grows, the radiation becomes larger but of course, the formula grows more inaccurate.

The open circuited coax in the example above was radiating out of a ground plane. We would like to know the power radiated from a coax without the ground plane. Since the coax is small we can assume that the cable does not mask the radiation which was below the ground plane. The field strength is the same with or without the ground plane. The energy is radiated over all space and not just half; therefore twice the power is radiated and the radiation conductance is twice

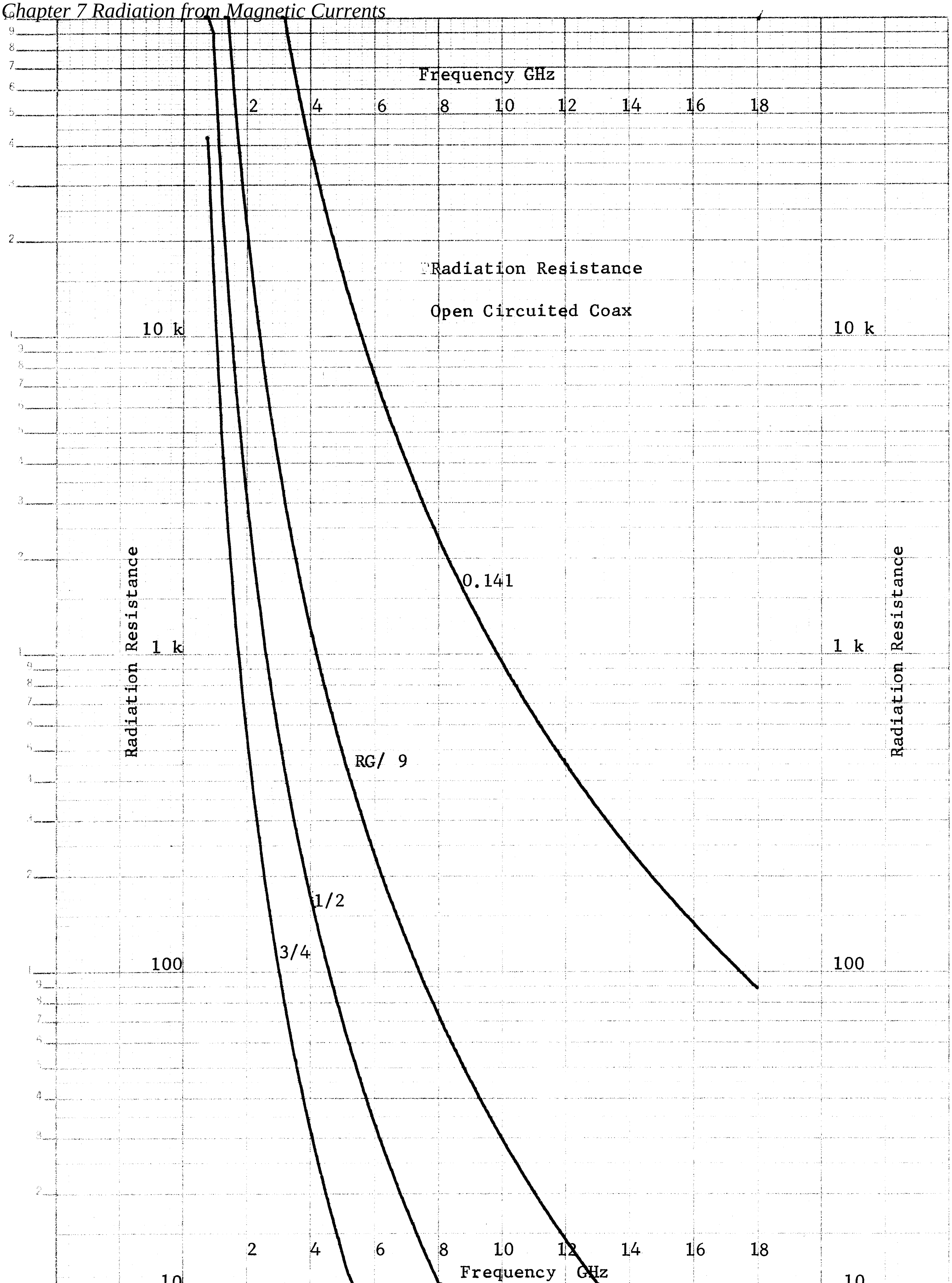
$$G_R = \frac{8\pi^5}{3\eta}\left[\frac{b^2 - a^2}{\lambda^2 \ln(b/a)}\right]^2$$

On page 197 is a plot of some common cable radiation resistances or "open circuit" impedances.

With this example we have come full circle. On page 174 a small current loop was analyzed as a magnetic current element. In the example above the small magnetic current loop has been analyzed as an electric current element. This is an example of duality. If we have a problem with only electric sources and the same problem with magnetic sources, then the solutions are the same with an interchange of the following quantities.

Electric Sources	Magnetic Sources
$\bar{E}$	$\bar{H}$
$\bar{H}$	$-\bar{E}$
$\bar{J}$	$\bar{M}$
$\bar{A}$	$\bar{F}$
$j\omega\epsilon$	$j\omega\mu$
$j\omega\mu$	$j\omega\epsilon$
β	β
η	$1/\eta$

This is due to the symmetry of Maxwell's equations and the vector potentials. Since the fields are linear we can divide the problem into two parts. One is part contains only electric sources, J, and the other only magnetic sources, M. When the solution for each part is found, then the total solution is the sum of the two separate solutions.


Frequency GHz
2 4 6 8 10 12 14 16 18
Radiation Resistance
Open Circuited Coax
10 k
10 k
0.141
Radiation Resistance
1 k
1 k
Radiation Resistance
RG/ 9
1/2
3/4
100
100
2 4 6 8 10 12 14 16 18
Frequency GHz
10
10

WIDE SLOT

We have only considered narrrow slots and ignored the retardation time across the slot which restricts the solutions to slots narrow in wavelengths. For a wide slot we will still assume that the electric field is constant across the slot.

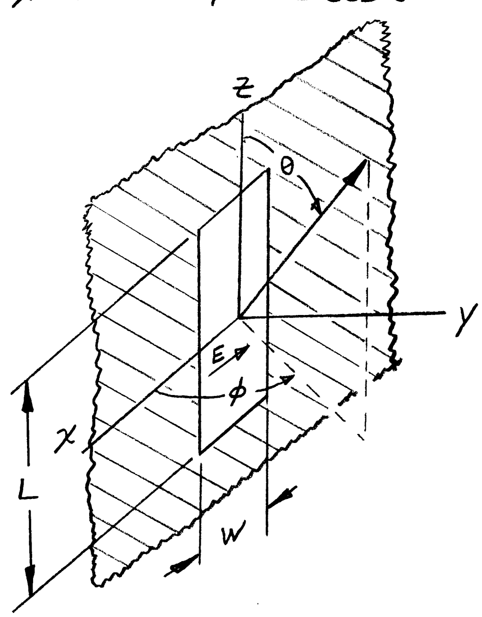
$$E_x = \frac{V_m}{W} \operatorname{Sin} \beta\left(\frac{L}{2} - |Z|\right) \qquad Y > 0$$

Using this tangential electric field, the equivalent magnetic current density can be found with the current backed by an electric conductor. The image of the horizontal magnetic current density is in the same direction. When the conductor is removed and the image source included, then the equivalent magnetic current density in free space becomes

$$\overline{M} = 2\,\overline{E} \times \hat{n} = 2\,E_x\,\overline{a}_z$$

When we integrate over this current density, we must find the distance to a zero phase reference plane for each direction of radiation. The center of the slot will be picked as the zero reference point of phase. For a given angle of radiation the distance from the source point to the zero reference plane is given as

$$x' \sin\theta \cos\phi + z' \cos\theta$$



When the far field electric vector potential is found, this term is used in the phase term and not in the amplitude 1/R term.

$$\overline{F} = \frac{e^{-j\beta r}\, 2 V_m}{4\pi r\, W} \int_{-W/2}^{W/2} \int_{-L/2}^{L/2} \sin\beta\left(\frac{L}{2} - |z|\right) e^{j\beta(x' \sin\theta\cos\phi + z\cos\theta)}\, dz\, dx$$

The integrand can be divided into two functions each one is a function of X or Z. When a double integral can be divided like this, then the integral is equal to the product of two integrals.

$$\int_0^a\int_0^b f(x)\,g(z)\,dx\,dz = \int_0^a g(z)\,dz \int_0^b f(x)\,dx$$

The integral expression for the electric vector potential is just this form. The integral can be expressed.

$$\bar{F} = \frac{e^{-j\beta r} V_m}{2\pi r W}\int_{-W/2}^{W/2} e^{j\beta x' \sin\theta\cos\phi}\,dx \int_{-L/2}^{L/2} \sin\beta\left(\frac{L}{2}-|z|\right)e^{jz\cos\theta}\,dz\;\bar{a}_z$$

The above separation of the integral into two integrals is a mathematical expression of pattern multiplication for apertures. Each integral taken separately gives the pattern of a linear array along each axis. The result of the two integrals is

$$F_z = \frac{e^{-j\beta r} V_m}{\pi\beta r}\left[\frac{\sin\beta\left(\frac{W}{2}\cos\phi\sin\theta\right)}{\beta\frac{W}{2}\cos\theta\sin\theta}\right]\left[\frac{\cos\left(\beta\frac{L}{2}\cos\theta\right)-\cos\left(\beta\frac{L}{2}\right)}{\sin^2\theta}\right]$$

Using this electric vector potential, the far field magnetic field is found from

$$H_\theta = j\omega\epsilon F_z \sin\theta \qquad \beta = \omega\sqrt{\mu\epsilon}$$

The magnetic field for the region $Y > 0$ is

$$H_\theta = \frac{j\,e^{-j\beta r} V_m}{\pi r \eta}\left[\frac{\sin\beta\left(\frac{W}{2}\cos\phi\sin\theta\right)}{\beta\frac{W}{2}\cos\phi\sin\theta}\right]\left[\frac{\cos\left(\beta\frac{L}{2}\cos\theta\right)-\cos\left(\beta\frac{L}{2}\right)}{\sin\theta}\right]$$

The electric field in the far field is found from the propagation of spherical waves in the positive R direction.

$$E_\phi = -\eta H_\theta$$

The total power radiated can be found by integrating the Poynting vector over one-eight of the radiation sphere and taking eight times this because of the symmetry of the fields around the slot.

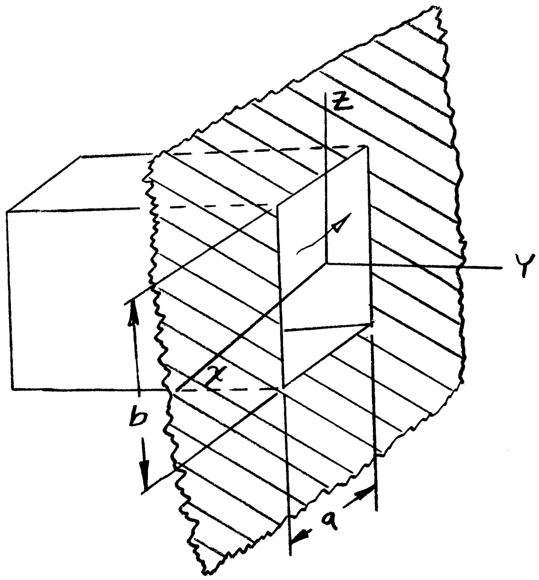
$$P_r = \frac{8|V_m|^2}{\pi^2\eta}\int_0^{\pi/2}\int_0^{\pi/2}\frac{\sin^2\left(\beta\frac{W}{2}\cos\phi\sin\theta\right)}{\beta^2\left(\frac{W}{2}\right)^2\cos^2\phi\sin^2\theta}\frac{\left[\cos\left(\beta\frac{L}{2}\cos\theta\right)-\cos\left(\beta\frac{L}{2}\right)\right]^2}{\sin\theta}\,d\theta\,d\phi$$

The radiation conductance of the slot is found from $G_r = \frac{P_r}{|V_m|^2}$

If the slot radiates only on one side, then the conductance will be one-half the value for the slot which radiates on both sides. The radiation conductance is

$$G_R = \frac{4\lambda^2}{\pi^4 \eta W^2} \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin^2\left(\beta \frac{W}{2}\cos\phi \sin\theta\right)\left[\cos\left(\beta\frac{L}{2}\cos\theta\right) - \cos\left(\beta\frac{L}{2}\right)\right]^2}{\cos^2\phi \sin^3\theta}\, d\theta\, d\phi$$

RADIATION FROM A WAVEGUIDE OPENING INTO A GROUND PLANE



Suppose we have a waveguide transmitting the dominant mode opening into a ground plane as shown in the figure above. If we assume that the aperture field is the same as in the waveguide, then the far field can be found. The aperture field is

$$E_x = E_o \cos\left(\frac{\pi z}{b}\right)$$

Using this tangential electric field, the equivalent magnetic current density can be found from

$$\bar{M} = 2\bar{E} \times \hat{n} = 2E_o \cos\left(\frac{\pi z}{b}\right) \bar{a}_z \qquad y > 0$$

The phase difference of each point on the aperture to the zero reference point is

$$\beta(x' \sin\theta \cos\phi + z' \cos\theta)$$

Using this magnetic current density, the electric vector potential can be found by integrating over the aperture plane. The far field electric potential is given by the following integral.

$$\bar{F} = \frac{2e^{-j\beta r}E_0}{4\pi r}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\cos\left(\frac{\pi z}{b}\right)e^{j\beta(x'\sin\theta\cos\phi + z'\cos\theta)}\,dx\,dz\;\bar{a}_z$$

This integral can be separated into the product of two integrals. The separation of the integrals is the mathematical expression of pattern multiplication of two linear arrays. The far field electric vector potential is

$$F_z = \frac{2E_0 b\sin\left(\frac{\beta a}{2}\sin\theta\cos\phi\right)\cos\left(\frac{\beta b}{2}\cos\theta\right)}{r\beta\sin\theta\cos\phi\left(\pi^2-(\beta b\cos\theta)^2\right)}$$

From this the far field magnetic field is found from $H_\theta = j\omega\epsilon F_z\sin\theta$
Using the expression $\beta=\omega\sqrt{\mu\epsilon}$, the magnetic field is

$$H_\theta = \frac{j2E_0 b\,e^{-j\beta r}\sin\left(\frac{\beta a}{2}\sin\theta\cos\phi\right)\cos\left(\frac{\beta b}{2}\cos\theta\right)}{\eta r\cos\phi\left(\pi^2-(\beta b\cos\theta)^2\right)} \qquad y>0$$

The electric field in the propagated spherical wave is found from: $E_\phi = \eta H_\theta$

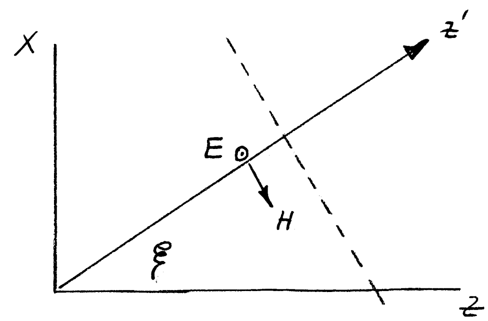
The total power radiated from the waveguide can be found by integrating the Poynting vector over a large sphere. It is difficult to relate this to a radiation conductance in the waveguide. The aperture field is the combination of the field incident on the aperture from the wave traveling down the guide and the reflected wave at the aperture. The aperture electric field is related to the electric field in the waveguide mode by:

$$E_0 = E_W(1+\Gamma)$$

where Γ is the voltage reflection coefficient of the aperture.

REFLECTION OF WAVES

The reflection of plane waves at a boundary will be considered. From the boundary conditions on page 153, we know that the tangential electric and magnetic fields are continuous across a boundary. Suppose an electromagnetic wave is propagating at an angle ξ with respect to the Y-Z plane with the electric field parallel to the Y axis.



The figure above shows the wave with the directions of the electric and magnetic fields. The coordinate, Z', in the direction of propagation is given as

$$z' = z\cos\xi + x\sin\xi$$

The unit vector in the X' direction is

$$\bar{a}_{x'} = \bar{a}_x\cos\xi - \bar{a}_z\sin\xi$$

The electric and magnetic fields are given by

$$E_y = E_0 e^{-j\beta z'} = E_0 e^{-j\beta(x\sin\xi + z\cos\xi)}$$

$$\bar{H} = -H_0(\bar{a}_x\cos\xi - \bar{a}_z\sin\xi)e^{-j\beta(x\sin\xi + z\cos\xi)}$$

Now suppose we have a wave traveling in the negative ξ direction as well. The equations for the electric and magnetic fields are given by

$$E_y = E_0 e^{-j\beta(-x\sin\xi + z\cos\xi)}$$

$$\bar{H} = -H_0(\bar{a}_x\cos\xi + \bar{a}_z\sin\xi)e^{-j\beta(-x\sin\xi + z\cos\xi)}$$

since $\sin(-\xi) = -\sin\xi$ and $\cos(-\xi) = \cos(\xi)$

If we take the superposition of these two waves with equal amplitudes and 180^o out of phase, then the two equations are combined to give an electric field.

$$E_y = A\left(e^{-j\beta x\sin\xi} - e^{j\beta x\sin\xi}\right)e^{-j\beta z\cos\xi}$$

We can combine this using the Euler identity $\sin U = \frac{1}{2j}(e^{jU} - e^{-jU})$

$$E_y = j2A \sin(\beta x \sin\xi)\, e^{-j\beta z \cos\xi}$$

Let us denote $j2A$ as E_o, $\beta_c = \beta \sin\xi$, and $\beta_g = \beta\cos\xi$. Then the equation for the combination of the two equal traveling waves is

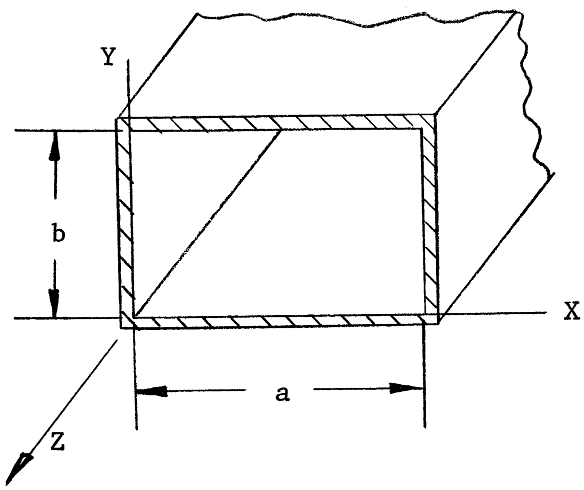
$$E_y = E_o \sin(\beta_c x)\, e^{-j\beta_g z}$$

The combination of waves traveling in ξ and $-\xi$ are a wave and its reflection from a boundary. A wave and its reflection have equal angles. Since the waves are out of phase at $X = 0$, the waves cancel on the boundary and satisfies the conditions for a conducting boundary (i.e. tangential electric field component equals zero).

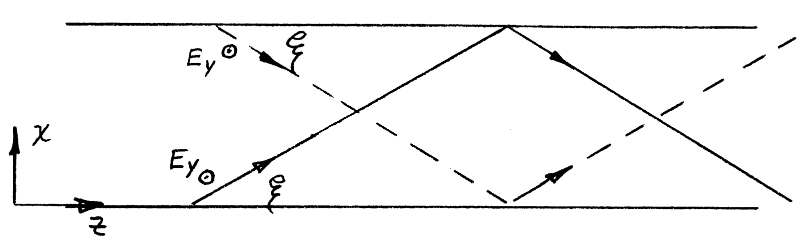
In the X direction we have a standing wave pattern. If the boundary is a dielectric boundary, then the tangential electric fields of the wave and its reflection will not cancel since there will be some wave in the dielectric. These reflections must be considered when designing radomes.

RECTANGULAR WAVEGUIDE

The rectangular waveguide is a hollow metal pipe with its cross section a rectangle.



We can use our solution for the combination of two waves traveling in the ξ and $-\xi$ directions with respect to the Y-Z plane to find the solution for the dominate mode in the waveguide. Looking straight down on the waveguide from above, we see two traveling waves in directions ξ and $-\xi$.



This will be a solution if the boundary conditions can be satisfied.

At a conducting boundary the tangential electric field must vanish. Our trial field is

$$E_y = E_o \sin(\beta_c x)\, e^{-j\beta_g z}$$

The electric field is only in the Y direction so the electric field has no tangential component on the top and bottom plates of the waveguide parallel to the X-Z plane in the figure. This field satisfies the boundary conditions on these plates.

The trial field has zero tangential component on the plane X = 0 since Sin(0) = 0. It only remains that the E_y field must be zero on the plane X = a. This is satisfied if

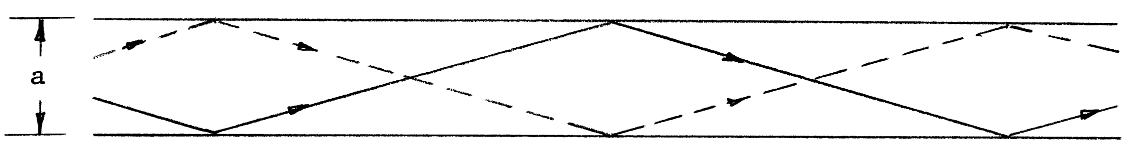
$$\beta_c a = n\pi \qquad n = 1, 2, 3, \ldots$$

The permissible values of β_c are called eigenvalues. The dominant mode is given for n = 1. When n and a are given, the direction of propagation of the two equal reflecting traveling waves is given

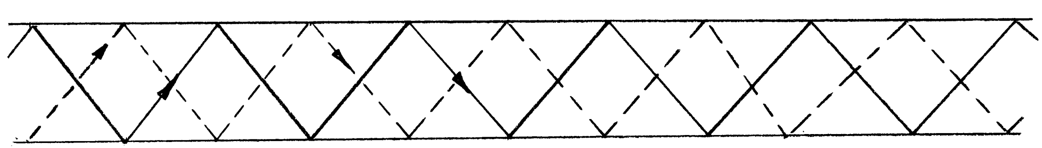
$$\beta \sin \xi = \frac{n\pi}{a} \qquad \beta = \frac{2\pi}{\lambda}$$

$$\xi = \sin^{-1}\left(\frac{n\lambda}{2a}\right)$$

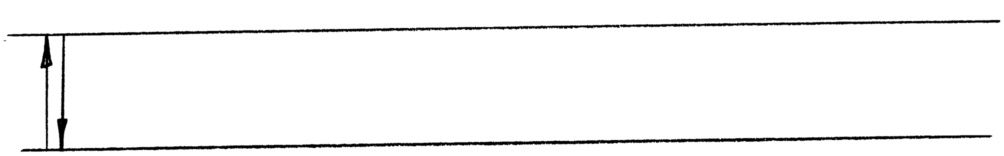
As the frequency increases the wavelength decreases and the angle decreases. If $n\lambda = 2a$, then $\theta = 90^o$ and the wave does not travel down the guide but reflects back and forth between the two walls X = 0 and X = a.



High Frequency



Low Frequency



Cut-off Frequency

This wavelength is called the cut-off wavelength of the waveguide. To propagate the dominant mode, n = 1, the frequency must be high enough so that the wavelength is less than 2 a, twice the width of the waveguide. The cut-off frequency of a waveguide in the dominant mode is given by

$$F_c\ (\text{GHz}) = 5.9014/(a\sqrt{\epsilon_r}) \qquad a \text{ - width (inches)}$$

The two waves which are reflecting back and forth between the walls are traveling at the velocity of light in the dielectric filling the waveguide. Since the waves must travel back and forth, the propagation of the waves in the Z direction is given by

$$\beta_g = \beta \cos \xi$$

We can find a relationship between the propagation constants in the X and Z directions by using a trigonometry identity

$$\beta^2(\sin^2\xi + \cos^2\xi) = \beta^2$$

$$\beta_c^2 + \beta_g^2 = \beta^2$$

$$\beta_g^2 = \beta_c^2 - \beta^2 = \left(\frac{\pi}{a}\right)^2 - \beta^2$$

The waves traveling in the waveguide have a wavelength associated with β_g

$$\beta_g = \frac{2\pi}{\lambda_g}$$

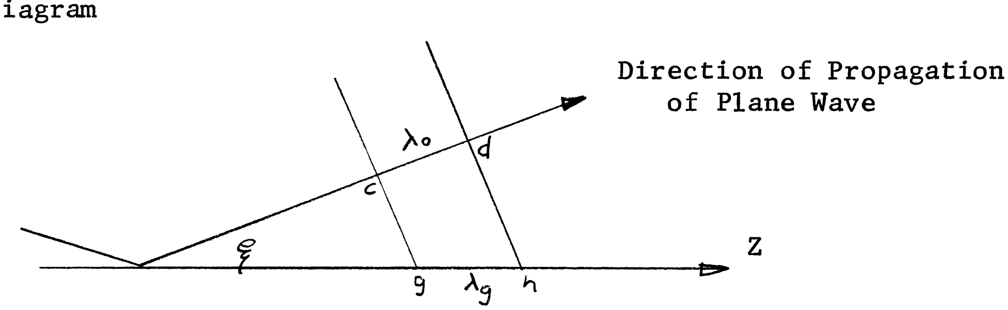
We can find the guide wavelength from these equations.

$$\lambda_g = \frac{\lambda_o}{\sqrt{1 - (\lambda_o/\lambda_c)^2}}$$

As the free space wavelength approaches the cut off wavelength of the guide, then the waveguide wavelength approaches infinity. The phase velocity of the waves is found from the equation.

$$v_g = f\lambda_g$$

Since the guide wavelength is larger than the free space wavelength, the phase velocity in the guide is larger than free space. This can be seen from the following diagram



When the phase changes as the wave travels from point c to d, the phase must change by the same amount traveling from g to h. The wave must travel faster to cover the larger distance in the same time. A waveguide is called a fast wave structure. The actual energy is reflecting back and forth between the two plates and still traveling at the free space velocity. Velocity of the energy or group velocity is less than the free space velocity where group velocity and phase velocity are equal.

CUT OFF WAVEGUIDE

When the wavelength exceeds the cut-off wavelength of the guide, then the hollow pipe will not propagate the energy in this mode. The propagation constant is complex for these frequencies and the wave is attenuated in the guide.

$$\alpha = \frac{2\pi}{\lambda_0 \lambda_c}\sqrt{\lambda_0^2 - \lambda_c^2} \qquad \text{nepers/length}$$

The propagation in the Z direction is $e^{-\alpha z}$,

The electric field is transverse to the direction of propagation and this mode is called the TE_{10} (Transverse Electric) mode.

$$E = E_0 \sin(\beta_c x)\, e^{-j\beta_g z}$$

The magnetic field can be found from Maxwell's curl equation.

$$\bar{H} = \frac{\nabla \times \bar{E}}{-j\omega\mu}$$

$$\nabla \times \bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$\nabla \times \bar{E} = -\bar{a}_x \frac{\partial}{\partial z} E_y + \bar{a}_z \frac{\partial}{\partial x} E_y$$

$$H_x = \frac{\beta_g E_0}{\omega\mu} \sin(\beta_c x)\, e^{-j\beta_g z}$$

$$H_y = \frac{\beta_c E_0}{-j\omega\mu} \cos(\beta_c x)\, e^{-j\beta_g z}$$

The wave impedance in the Z direction is given by

$$Z_z = \frac{E_y}{H_x} = \frac{\omega\mu}{\beta_g} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} \qquad f > f_c$$

In a waveguide both the impedance and phase velocity are a function of frequency. A transmission line with these properties is called dispersive.

The waveguide is only useful until the next higher order mode can propagate in the guide. Most of the commercially available waveguide has a height about one-half the width. In that case the next higher order mode is TE_{20} or TE_{01} mode. The waves that are reflecting back and forth in the TE_{20} mode are traveling a full wavelength between the side walls. The guide must have a width greater than one wavelength for the wave to propagate in the TE_{20} mode. In the TE_{01} mode the waves are traveling back and forth between the top and bottom walls; across the narrow dimension of the guide. Every waveguide mode can be described as two waves reflecting back and forth in the guide as it propagates. The general TE_{mn} mode waves are at an angle to both the sets of walls of the rectangular waveguide.

We could be perfectly happy only considering the electromagnetic waves reflecting back and forth between the parallel plates across the width of the waveguide. These waves induce currents in the walls of the waveguide. We will need the distribution of currents to discuss slots cut in the wall of the waveguide. The currents can be found from the tangential magnetic fields and applying the boundary condition given on page 154; the surface current density is

$$\bar{J}_s = \hat{n} \times \bar{H}$$

On the side walls $\hat{n} = \bar{a}_x$ for $x = 0$. The magnetic field at $X = 0$ is

$$\bar{H} = \frac{\beta_c E_o}{-j\omega\mu} e^{-j\beta_g z} \bar{a}_z$$

There is only a Z component at $X = 0$. The surface current is

$$J_y = \frac{\beta_c E_o}{j\omega\mu} e^{-j\beta_g z}$$

For $X = a$, $\hat{n} = -\bar{a}_x$ and $\cos(\beta_c a) = -1$, therefore the magnetic field is

$$\bar{H} = \frac{\beta_c E_o}{j\omega\mu} e^{-j\beta_g z} \bar{a}_z$$

From the vector cross product $-\bar{a}_x \times \bar{a}_z = \bar{a}_y$, the current on the wall $X = a$ is

$$\bar{J}_s = \frac{\beta_c E_o}{j\omega\mu} e^{-j\beta_g z} \bar{a}_y$$

The currents on the side walls are in the direction of the height on the waveguide and are traveling waves in the direction of propagation.

On the top and bottom walls the normal vector is $-a_y$ and a_y, respectively. The surface currents on these walls are

$$\bar{J}_s = \bar{a}_y \times (H_x \bar{a}_x + H_z \bar{a}_z)$$

$$\bar{J}_s = -\bar{a}_z H_x + \bar{a}_x H_z \qquad \text{lower wall}$$

$$\bar{J}_s = \bar{a}_z H_x - \bar{a}_x H_z \qquad \text{upper wall}$$

$$\bar{J}_s = \frac{-\beta_c E_0}{\omega\mu}\left[\frac{\cos(\beta_c x)\,\bar{a}_x}{j} + \sin(\beta_c x)\,\bar{a}_z\right] e^{-j\beta_g z} \quad \text{lower wall}$$

$$\bar{J}_s = \frac{\beta_c E_0}{\omega\mu}\left[\frac{\cos(\beta_c x)\,\bar{a}_x}{j} + \sin(\beta_c x)\,\bar{a}_z\right] e^{-j\beta_g z} \quad \text{upper wall}$$

Note that the currents in the X direction are 90° out of phase with respect to the electric field. The field in the X direction is a standing wave while the currents in the Z direction are in phase with the electric field (voltage) which is true for a propagating wave. The X directed currents on the top and bottom walls match the side wall currents and are zero in the center of the guide. The Z directed currents on the top and bottom walls are equal and opposite and correspond to the currents on a two wire transmission line. These Z directed currents are a maximum at the center of the guide and taper to zero at the side walls.

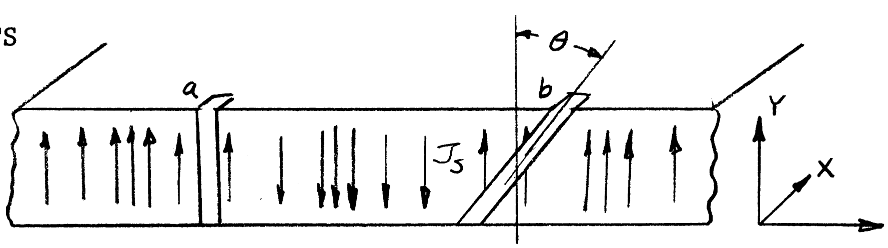
SLOTS IN RECTANGULAR WAVEGUIDE

When we consider slots cut in the walls of the waveguide, we will make the following assumptions.

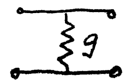
1). The slot is narrow
2) The slot is resonant; length $\simeq \lambda/2$
3) The field in the slot is transverse to the long dimension and varies sinusoidally along the slot and is independent of the exitation system.
4) The waveguide walls are perfectly conducting and infinitely thin. Even through the walls are not infinitely thin the differences between this idealized guide and real guides are small.

The slot is excited by cutting lines of current in the walls of the guide. If the slot does not cut the lines of current in the walls, then the slot does not load the waveguide transmission line and there is not power transfer to the slot. The slot can cut two types of wall currents. If the slot cuts Z directed currents, then the slot will appear as a series load to the waveguide. Currents in the X or Y directions are shunt currents and slots which cut these currents present shunt loads to the waveguide transmission line.

SIDE WALL SLOTS



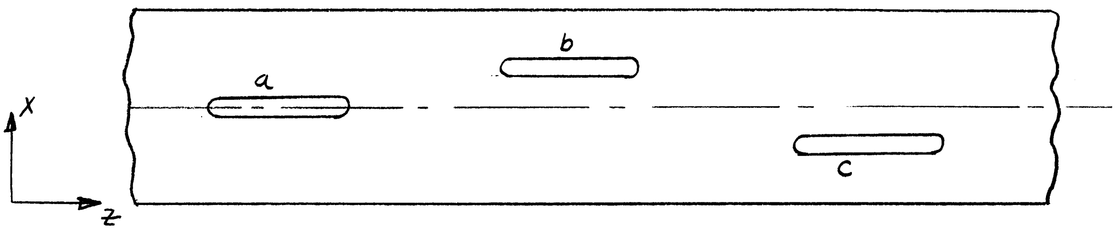
The currents in the side walls are only in the Y direction and are shunt currents. Any slot which cuts these currents are shunt loads. The slot a in the figure does not cut any surface currents and is not excited by the waveguide. The slot b cuts the surface currents and is excited by the waves in the waveguide. The shunt load the slot loads the guide is



$$g = k \sin^2(\theta)$$

The load is proportional to the square of the sine of the angle with respect to the Y axis. This is the only possible side wall slot.

TOP WALL LONGITUDINAL SLOT

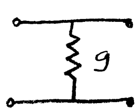


The longitudinal slot can only cut currents in the X direction. The X directed currents are shunt currents between the top and bottom walls; therefore the longitudinal slot is a shunt load to the waveguide. The X directed currents are

$$J_x = J_0 \cos(\beta_c x) \qquad \beta_c = \frac{\pi}{a}$$

$$J_x = J_0 \cos\left(\frac{\pi x}{a}\right)$$

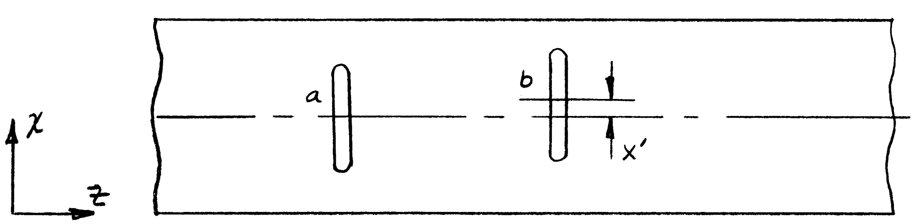
In the center of the guide, X = a/2 and the X directed surface current is zero. The slot a in the figure above is not excited because the currents are zero in the center of the guide wall. Note this is the slot location used for a waveguide slotted line and is not excited by the traveling waves in the guide. As the slot is moved off the center of the broadwall, it cuts currents and is excited. Slots b and c in the figure above are excited. The two slots b and c cut currents with different signs in the X direction and are excited 180^{o} out of phase. The equivalent circuit of this slot is



$$g = k \cos^2\left(\frac{\pi x}{a}\right)$$

where X is the distance from the edge of the guide.

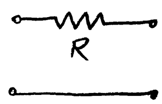
TOP WALL TRANSVERSE SLOT



This slot will cut Z directed currents and will be a series load to the waveguide. The Z directed current in the top wall varies as

$$J_z = J_o \sin\left(\frac{\pi x}{a}\right)$$

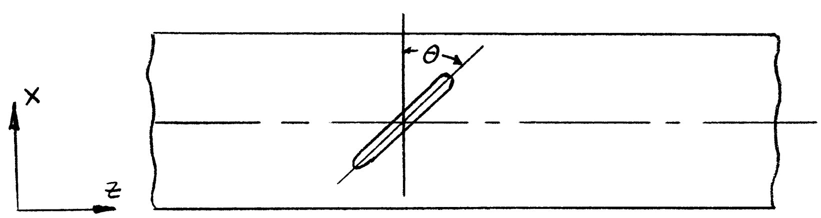
The slot a in the figure above is centered in the top wall and cuts more surface current lines than the slot b which is offset from the center by a distance X_1. The center slot presents a larger series load to the wave-guide and is excited more than the offset slot. The equivalent waveguide circuit element of the slot is



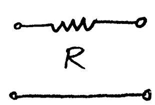
$$R = R_0 \cos\left(\frac{\pi X_1}{a}\right)$$

where X_1 is measured from the center of the guide.

INCLINED BROADWALL SLOTS



If a slot is cut in the broadwall slot centered and at an angle with respect to the X axis, the slot will be excited when $\theta < 90^o$. The centered slot will cut X directed currents on both sides of the center line, but since the currents are 180^o out of phase on the two sides of the center line, there will be no net shunt current cut by the slot. The slot has no shunt loading to the waveguide. The slot also cuts longitudinal currents. The amount of longitudinal current lines cut depends on the cosine of the angle θ. Since the slot cuts only Z directed currents, it is a series load to the waveguide transmission line.



$$R = R_0 \cos^2\theta$$