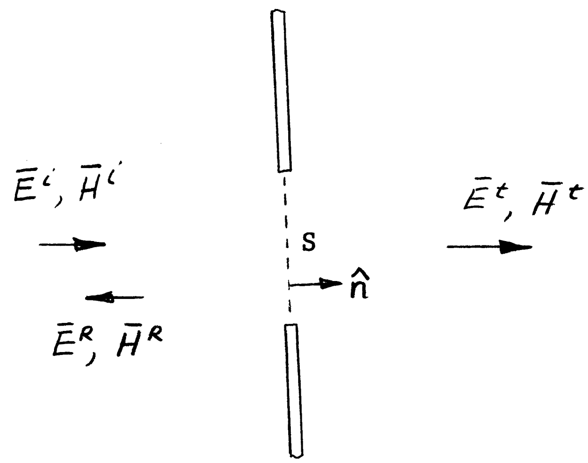
INDUCTION THEOREM

Suppose we have a screen with an aperture and an incident field on the screen from the left. There will be fields transmitted through the screen



to the right, E^t, H^t, and fields reflected by the screen, E^r, H^r. The total fields on the left of the screen is the sum of the incident and reflected fields.

$$E = E^i + E^r \qquad H = H^i + H^r$$

Over the surface, S, the tangential components of the fields must be continuous because there is no medium to support any currents or charges.

$$E^t_{tan} = E^i_{tan} + E^r_{tan} \qquad H^t_{tan} = H^i_{tan} + H^r_{tan}$$

Similiarly the normal components are continuous across the aperture because there are no trapped charges on the boundary.

Now let us define a scattered field as the combination of the reflected fields to the left of the screen and the transmitted fields to the right of the screen. We can generate this scattered field by a distribution of sources over the aperture in the screen. This scattered field is discontinuous in the aperture. The discontinuity in the tangential electric field is equal to a magnetic surface current density given by the boundary condition:

$$\bar{M}_s = (\bar{E}^t - \bar{E}^r) \times \hat{n}$$

given on page 175. The tangential magnetic field discontinuity is equal to an electric surface current density:

$$\bar{J}_s = \bar{n} \times (\bar{H}^t - \bar{H}^r)$$

from the boundary condition on page 154. These currents are the virtual sources of the scattered field.

We can find the difference of the reflected and transmitted field from the continuity of the fields across the aperture boundary.

$$E^t - E^r = E^i \qquad H^t - H^r = H^i$$

Using these equations, we can find the virtual sources of the scattered field from the incident wave by combining the above sets of equations.

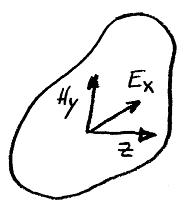
$$\bar{M}_s = \bar{E}^i \times \bar{n} \qquad \bar{J}_s = \bar{n} \times \bar{H}^i$$

These virtual sources are a function of the incident wave and they radiate in the presence of the aperture screen. In general the problem is still a boundary value problem to find the scattered field, but we can approximate the scattered field by using the Huygens source approximation. Fortunately, the radiation scattered toward the aperture is small and we can find reasonable approximations in the direction normal to the aperture.

HUYGENS SOURCE

The Huygens source principle states that every point on a given wavefront can be considered as a secondary source which gives rise to a spherical wavelet. The wave at a distance field point can be obtained as a superposition of these elementary wavelets. We will use the induction theorem to define these sources more precisely.

Suppose we have an aperture with an incident field with only E_x and H_y components. Using the induction theorem we can define virtual sources on



the aperture which can be used to find the transmitted fields. The equivalent magnetic surface current is

$$\bar{M}_s = \bar{E}^i \times \hat{n} = E_x \bar{a}_x \times \bar{a}_z = -E_x \bar{a}_y$$

Similiarly, we can find the equivalent electric surface current for the magnetic field.

$$\bar{J}_s = \bar{n} \times \bar{H}^i = H_y \bar{a}_z \times \bar{a}_y = -H_y \bar{a}_x$$

The radiation from these virtual surface currents can be found by using the magnetic and electric vector potentials. The differential electric vector potential is found from the magnetic surface currents.

$$\bar{F}\, dx\, dy = \frac{e^{-j\beta r}}{4\pi r} E_x\, dx\, dy\, \bar{a}_y$$

The far field magnetic field is found from this as $\bar{H} = -j\omega\epsilon\bar{F}$.

$$H_\theta = -j\omega\epsilon F_y\, \bar{a}_\theta \cdot \bar{a}_y = -j\omega\epsilon F_y \cos\theta \sin\phi$$

The differential magnetic field due to the differential source is

$$H_\theta\, dx\, dy = \frac{j\omega\epsilon}{4\pi r} E_x \cos\theta \sin\phi\, dx\, dy$$

The electric field from this source is $E_\phi = -\eta H_\theta$; substituting $\eta\omega\epsilon = \frac{2\pi}{\lambda}$ we get the differential electric field.

$$E_\phi\, dx\, dy = \frac{-j e^{-j\beta r}}{2\lambda r} E_x \cos\theta \sin\phi\, dx\, dy$$

Similiarly, the H_ϕ component can be found and the E_θ field from that as $E_\theta = \eta H_\phi$, then

$$H_\phi\, dx\, dy = -j\omega\epsilon F_y\, \bar{a}_\phi \cdot \bar{a}_y\, dx\, dy$$

$$H_\phi\, dx\, dy = \frac{j\omega\epsilon E_x e^{-j\beta r}}{4\pi r} \cos\phi\, dx\, dy$$

$$E_\theta\, dx\, dy = \frac{j e^{-j\beta r}}{2\lambda r} E_x \cos\phi\, dx\, dy$$

The radiation found from the magnetic vector potential due to electric surface currents also adds to the total field. The differential magnetic vector potential is

$$\bar{A}\, dx\, dy = \frac{-e^{-j\beta r}}{4\pi r} H_y\, dx\, dy\, \bar{a}_x$$

The far field electric field is found from this as $\bar{E} = -j\omega\mu\bar{A}$.

$$E_\theta = -j\omega\mu A_x \bar{a}_\theta \cdot \bar{a}_x \qquad E_\phi = -j\omega\mu A_x \bar{a}_\phi \cdot \bar{a}_x$$

The differential far field electric fields due to the differential surface current is

$$E_\theta\, dx\, dy = \frac{j\omega\mu\, e^{-j\beta r}}{4\pi r} H_y \cos\theta \cos\phi\, dx\, dy$$

$$E_\phi\, dx\, dy = -\frac{j\omega\mu\, e^{-j\beta r}}{4\pi r} H_y \sin\phi\, dx\, dy$$

One of the key elements of the Huygens source approximation is to assume that the fields in the aperture are free space waves so that the ratio of the electric field to the magnetic field is equal to the impedance of free space. This will be true for apertures which are at least a few wavelengths across. The Huygens source method will give approximate results even when this is not true. With this approximation we can replace the magnitude of the magnetic field in the aperture with the electric field divided by the impedance of free space. If we also note that $\omega\mu/\eta = 2\pi/\lambda$ then the electric field from the electric current sources can be replaced with:

$$E_\theta \, dx\, dy = j\frac{e^{-j\beta r}}{2\lambda r} E_x \cos\theta \cos\phi \, dx\, dy$$

$$E_\phi \, dx\, dy = -j\frac{e^{-j\beta r}}{2\lambda r} E_x \sin\phi \, dx\, dy$$

When we combine the electric fields from both types of differential sources, we have the total differential electric field in the far field.

$$E_\theta \, dx\, dy = j\frac{e^{-j\beta r}}{2\lambda r} E_x (1+\cos\theta)\cos\phi \, dx\, dy$$

$$E_\phi \, dx\, dy = -j\frac{e^{-j\beta r}}{2\lambda r} E_x (1+\cos\theta)\sin\phi \, dx\, dy$$

These equations are the differential electric field Huygens source fields. We can find the total field by integrating this over the aperture. The magnetic field can be found from the propagation restriction, $|\bar{H}| = |\bar{E}|/\eta$.

PROPERTIES OF HUYGENS SOURCE

Polarization The source has both θ and ϕ components for an X directed E field in the aperture. The far field X and Y components are found from

$$E_X = E_0 (\cos\phi \; \bar{a}_x \cdot \bar{a}_\theta - \sin\phi \; \bar{a}_x \cdot \bar{a}_\phi)$$

$$E_X = E_0 (\cos^2\phi \cos\theta + \sin^2\phi)$$

$$E_y = E_0 (\cos\phi \; \bar{a}_y \cdot \bar{a}_\theta - \sin\phi \; \bar{a}_y \cdot \bar{a}_\phi)$$

$$= E_0 (\cos\phi \sin\phi \cos\theta - \sin\phi\cos\phi)$$

$$= E_0 (\cos\theta - 1)\cos\phi \sin\phi$$

$$@\ \theta = 0 \qquad E_x = E_0 \qquad E_y = 0 \qquad\qquad E_0 = j\frac{e^{-j\beta r}}{2\lambda r} E_{x0} (1+\cos\theta)$$

Obliquity Factor The Huygens source fields have the factor $(1 + \cos\theta)/2$ which is called the obliquity factor. This is the same as $\cos^2(\theta/2)$. The source does not radiate equally in all directions but has a pattern.

Phase Shift   The curious thing is the extra phase shift of the Huygens source. There is an extra 90° phase shift in the differential source term. It has been said that this is due to the extra velocity of a spherical wave at its source.

Restrictions   We must remember the restrictions due to the assumptions. These are:

1) The aperture is wide enough that $|\bar{E}| = \eta |\bar{H}|$ in the aperture. For smaller apertures we could use the Z directed impedance.

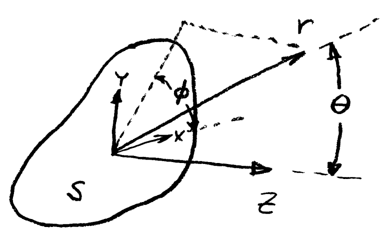
2) The proper boundary conditions have not been applied to give zero fields on the material around the aperture. This restricts the field solution to θ near boresight. The method will not predict more than a few sidelobes.

APERTURES

We will use the Huygens source to analyze the radiation from an aperture. Suppose we have an aperture in the X-Y plane with a known incident field. Also assume that the aperture is large enough for the Huygens source restrictions to be satisfied, i.e. $|E| = \eta |H|$, in the aperture. An aperture is a continuous array in two dimensions. For each direction of propagation we will need a zero reference plane which is used to find the phase difference of each point to the far field point. By the radiation approximation, it is assumed that the distance from each source point to the far field point is the same for all points on the aperture for amplitudes. We will have the zero reference plane through the origin of the coordinate system; then the distance from a source point to the reference plane is

$$d = x' \sin\theta \cos\phi + y' \sin\theta \sin\phi + z' \cos\theta$$

where (θ, ϕ) is the direction of radiation and (X', Y', Z') is the source point.



Assume that the field incident on the aperture from the negative Z axis is linearly polarized with the electric field aligned with the X axis, then the far field electric field is

$$E_\theta = \frac{j e^{-j\beta r}}{2\lambda r}(1+\cos\theta)\cos\phi \iint_S E_0(x,y)\, e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx\,dy$$

$$E_\phi = \frac{-j e^{-j\beta r}}{2\lambda r}(1+\cos\theta)\sin\phi \iint_S E_0(x,y)\, e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx\,dy$$

Where $E_o(X,Y)$ is the complex excitation of the aperture (amplitude and phase) and it is assumed $Z' = 0$. The integration is over the aperture surface, S. The terms (1 + Cos(Theta)), Cos(Phi), and Sin(Phi) can be removed from the integrals because they are constant over the arguments of the integrals. The principle plane patterns are

E_θ at $\phi = 0$ (E Plane)

E_ϕ at $\phi = 90$ (H Plane)

If we ignore the sign change in the patterns going from E_θ to E_ϕ, we find the general principle plane far field electric field as

$$E(\theta,\phi) = \frac{e^{-j\beta r}}{2\lambda r}(1+\cos\theta)\iint_S E_o(x,y)\, e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx\, dy$$

Using this electric field, we find the magnetic field from $|H| = |E|/\eta$.
The radiation intensity is found from

$$U(\theta,\phi) = \frac{(1+\cos\theta)^2}{4\lambda^2\eta}\left|\iint_S E_o(x,y)\, e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx\, dy\right|^2$$

Since $U(\theta,\phi) = |E(\theta,\phi)|^2/\eta$

GAIN

We can find the gain of the aperture if we can find the total power radiated by the aperture. The total power can be found by integrating the far field pattern over a large sphere. This is a problem because the far field pattern given above is not accurate everywhere because of the Huygens source approximation. In fact we can use one of the approximations to find the total power radiated. We have assumed that the electric and magnetic fields are related by the characteristic impedance of free space in the aperture. The total power is the integral of the Poynting vector magnitude over the aperture.

$$P_r = \frac{1}{\eta}\iint_S |E_o(x,y)|^2\, dx\, dy$$

From this the directivity is found from

$$\text{Directivity} = \frac{4\pi U_{MAX}}{P_r}$$

Substituting the equation for the radiation intensity, we find the directivity.

$$\text{Directivity} = \frac{\pi(1+\cos\theta)^2\left|\iint_S E_o(x,y)\, e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx\, dy\right|^2_{MAX}}{\lambda^2\iint_S |E_o(x,y)|^2\, dx\, dy}$$

If the maximum radiation intensity is at $\theta = 0$, then the directivity is simplified because $\sin\theta \to 0$ and the exponential term is removed.

$$\text{Directivity} = \frac{4\pi}{\lambda^2}\,\frac{\left|\iint_s E_0(x,y)\,dx\,dy\right|^2}{\iint_s |E_0(x,y)|^2\,dx\,dy}$$

Suppose $E_o(X,Y)$ = Constant, then the integral reduces to the area of the aperture.

$$\text{Directivity} = \frac{4\pi}{\lambda^2}\,\frac{|E_0 A|^2}{E_0^2 A} = \frac{4\pi A}{\lambda^2}$$

This is the expression for the Directivity (Gain) of a large uniformly fed aperture given on page 48. Since most large aperture radiators have small I^2R and reflection losses, directivity is taken for gain (such as horns).

PATTERN MULTIPLICATION

The aperture can be considered a continuous array of continuous arrays under special conditions. If the electric field excitation of the aperture, $E_o(X,Y)$, can be divided into the product of functions each of which is a function of only X or Y and the integral limits are constant X and Y coordinates, then the aperture can be analyzed as an array of arrays.

$$E_o(X,Y) = E_1(X)\,E_2(Y)$$

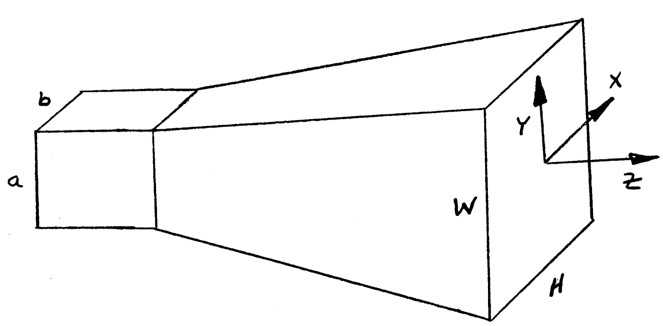
The integral for the radiation intensity becomes:

$$U(\theta,\phi) = \frac{(1+\cos\theta)^2}{4\lambda^2\eta}\left|\int E_1(x)\,e^{j\beta(x'\sin\theta\cos\phi)}dx \int E_2(y)\,e^{j\beta y'\sin\theta\sin\phi}dy\right|^2$$

The integral is separable.

RECTANGULAR HORN

We will use the Huygens source aperture method to analyze the radiation from rectangular horns fed from a rectangular waveguide in the dominate TE_{10} mode.

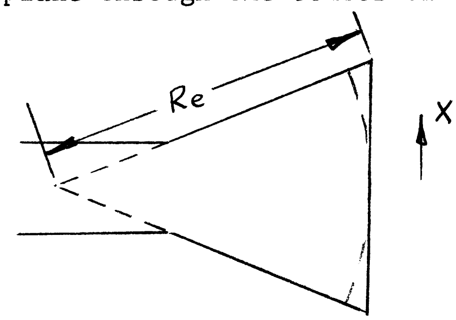


It is assumed that the horn tapers slowly from the waveguide to the mouth of the horn so that the incident field has the same mode pattern as the mode in the waveguide. The transition from the straight guide to the flared horn must generate higher order modes to satisfy the boundary conditions but we will assume that these are insignificant.

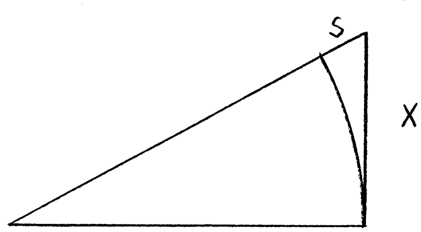
Assuming the electric field in the waveguide is polarized in the X direction, then the amplitudes in the two planes are:

E Plane	Constant
H Plane	$\cos\left(\frac{\pi Y}{W}\right)$

We have the amplitude of the excitation electric field but not the phase. Consider the E plane through the center of the horn. If we project the two



slant sides of the horn back into the feeding waveguide, they will intersect. The radius from this intersection to the outer edge is called the E Plane radius of the horn, R_e. The waves traveling to the outside of the horn must travel farther to the aperture plane than those traveling to the center of the horn aperture. The distance is a quadratic function of the coordinate X.



$$S = R_e - \sqrt{R_e^2 - X^2} = R_e\left(1 - \sqrt{1 - X^2/R_e^2}\right)$$

We can expand the square root in a Taylor series about X = 0.

$$\sqrt{1 - \frac{X^2}{R_e^2}} \simeq 1 - \frac{X^2}{2R_e^2}$$

$$S = \frac{X^2}{2R_e}$$

If we normalize S by dividing by the wavelength and let $X = H/2$, then S is called the maximum aperture phase deviation in wavelengths

$$S_e = \frac{H^2}{8 R_e \lambda}$$

The H Plane flare leads to a similiar phase deviation. The electric field in the aperture plane is

$$E_0(X,Y) = \cos\left(\frac{\pi Y}{W}\right) e^{-j\frac{\pi X^2}{\lambda R_e}} e^{-j\frac{\pi Y^2}{\lambda R_m}}$$

The integrand of the field integral is separable and limits of the integral are over constant X and Y coordinates; therefore the integral is separable into the product of two integrals.

$$u(\theta,\phi) = \frac{(1+\cos\theta)^2}{4\lambda^2\eta} \left| \int_{-W/2}^{W/2} \cos\left(\frac{\pi Y}{W}\right) e^{j\frac{2\pi}{\lambda} Y \sin\theta \sin\phi} e^{-j\frac{\pi Y^2}{\lambda R_m}} dy \right.$$

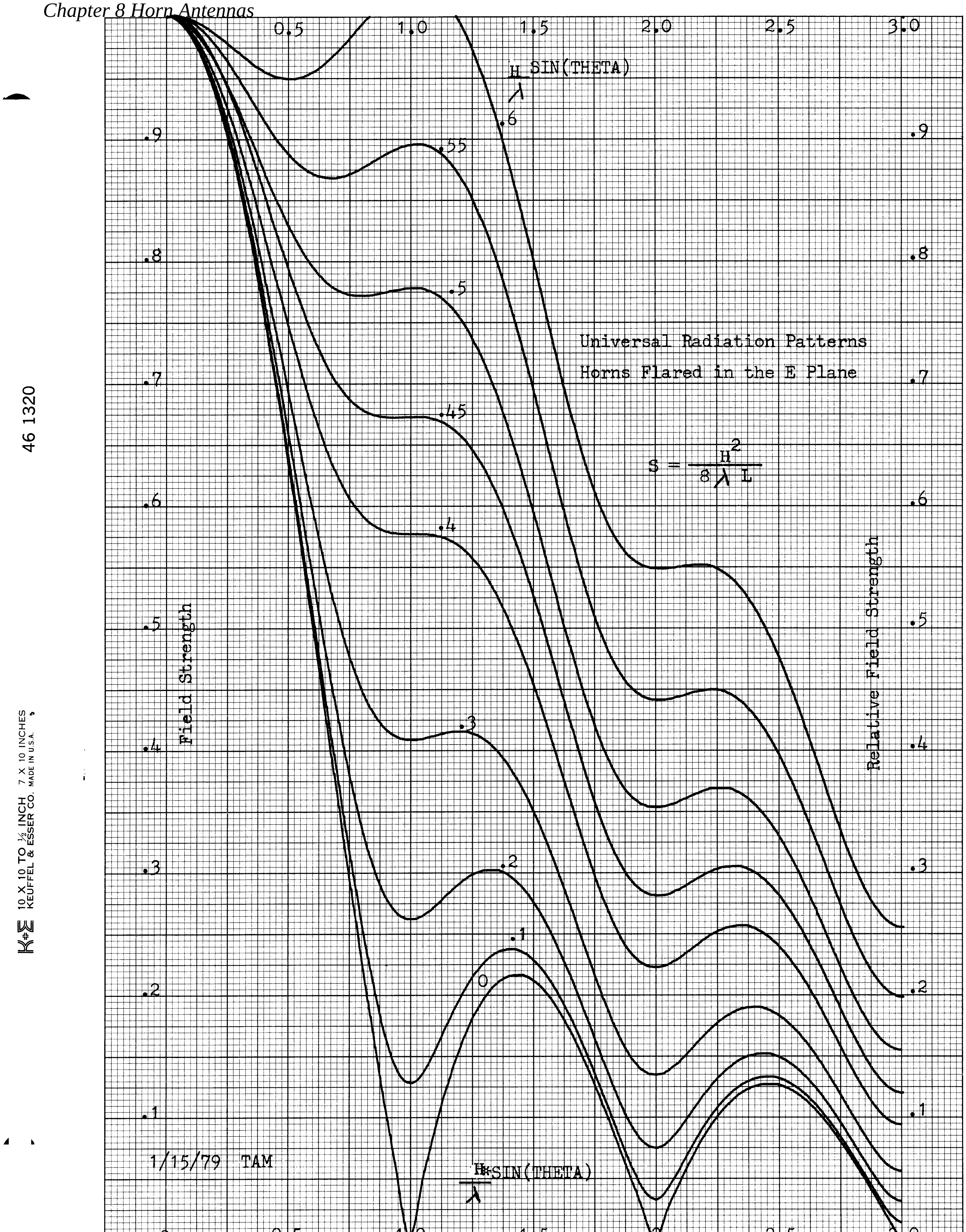
$$\left. * \int_{-H/2}^{H/2} e^{j\frac{2\pi}{\lambda} X \sin\theta \cos\phi} e^{-j\frac{\pi X^2}{\lambda R_e}} dx \right|^2$$

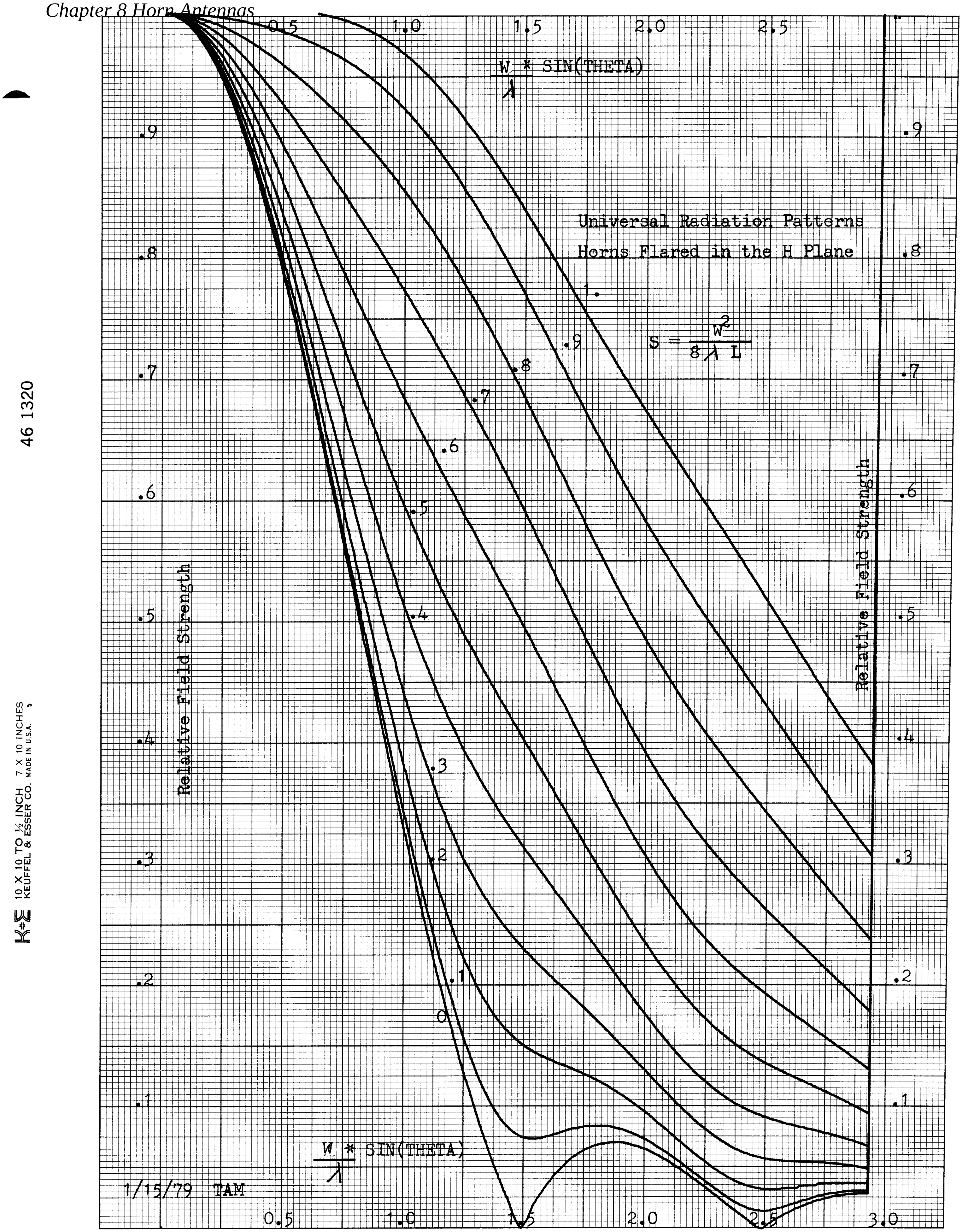
If we take the special cases of E plane ($\phi = 0$) and H plane ($\phi = 90^\circ$), we can plot universal radiation patterns for these two planes. The E and H plane universal radiation patterns for the TE_{10} mode rectangular horn are plotted on pages 220 and 221. S, the maximum phase deviation of the aperture in wavelengths, is the parameter of these curves. The factor $(1 + \cos(\theta))/2$ has been removed. The abscissa of the E plane universal pattern is $\frac{H \sin\theta}{\lambda}$, where H is the E plane height of the horn.

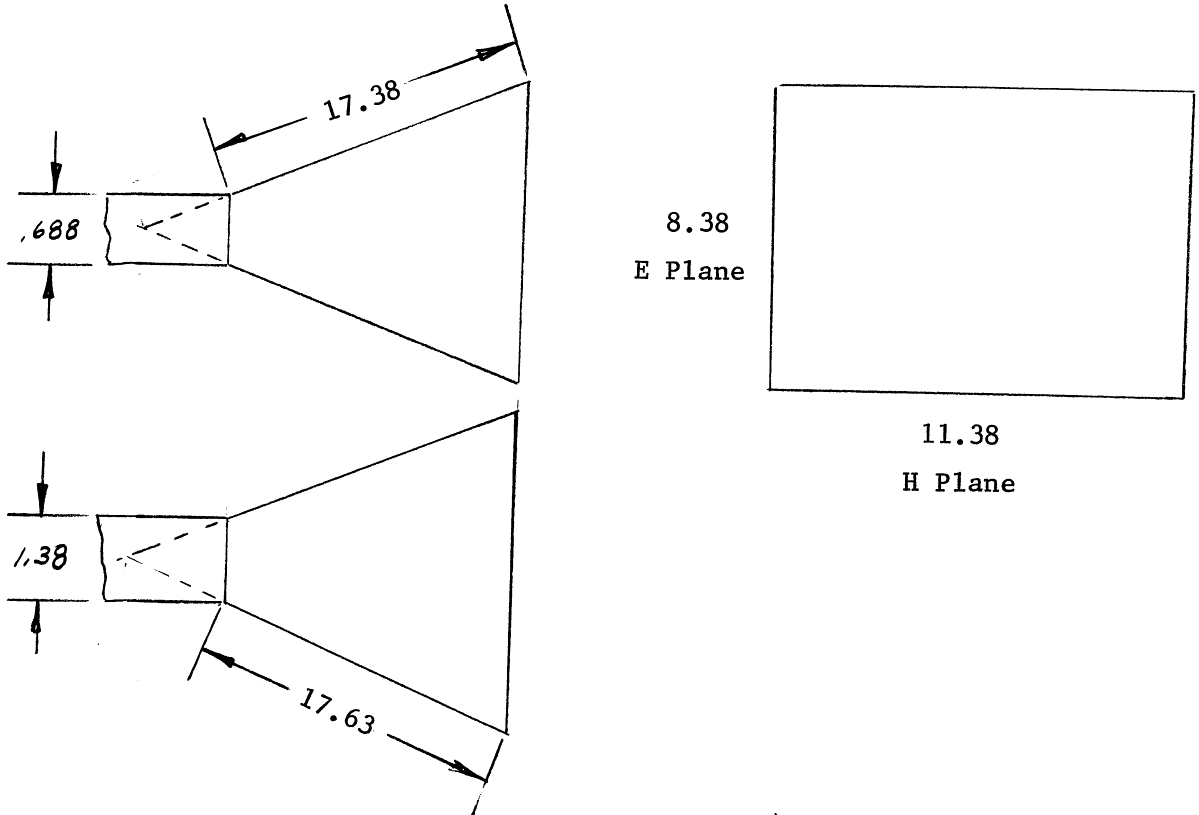
Similiarly, the abscissa of the H plane universal pattern is $\frac{W \sin\theta}{\lambda}$, where W is the H plane width of the horn. The ordinate of each plot is field strength or the electric field. The 3 dB point corresponds to 0.707 on the chart. These charts can be used to find the pattern of the horn.

The pattern in the E plane is that of a uniformly fed continuous array with a quadratic phase error. For large horns in terms of wavelengths, the $(1 + \cos(\theta))/2$ term can be ignored, but for smaller apertures this term must be multiplied by the value from the chart. The H plane pattern is that of a continuous array with a Cosine aperture illumination. Notice that the H plane pattern has smaller sidelobes than the E plane pattern. Also the H plane radius can be reduced further before the pattern splits on boresight as the E plane pattern does at $S_e = 0.6$.

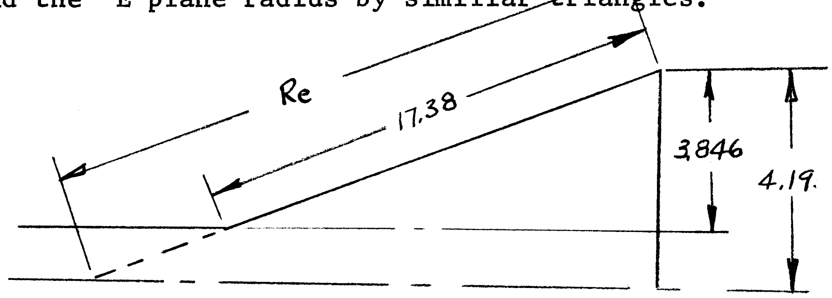
Let us use an actual horn to demonstrate the use of these curves. The following are the dimensions of the horn.


0.5
1.0
1.5
2.0
2.5
3.0
H/λ SIN(THETA)
.6
.55
.5
.45
.4
.3
.2
.1
0
.9
.8
.7
.6
.5
.4
.3
.2
.1
Universal Radiation Patterns
Horns Flared in the E Plane
S = H^2 / 8 λ L
Field Strength
Relative Field Strength
1/15/79 TAM
H/λ SIN(THETA)
0
0.5
1.0
1.5
2
2.5
3.0
46 1320
K+E 10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


W/λ * SIN(THETA)
Universal Radiation Patterns
Horns Flared in the H Plane
$S = \frac{W^2}{8 \lambda L}$
Relative Field Strength
Relative Field Strength
1/15/79 TAM
0.5
1.0
1.5
2.0
2.5
3.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
1.
0
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

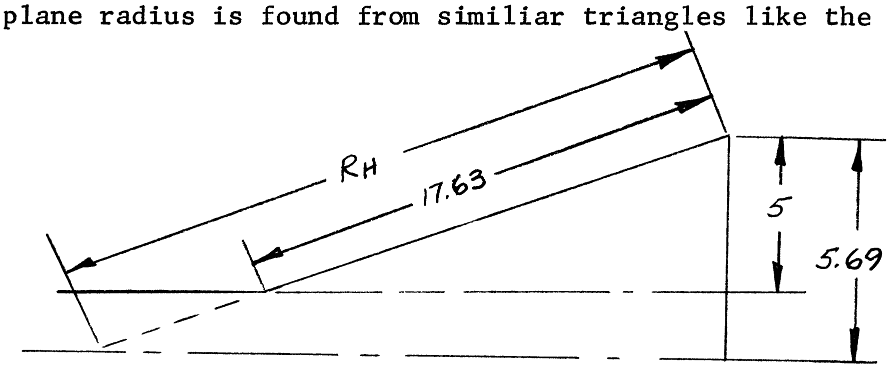


We can find the E plane radius by similiar triangles.



$$\frac{R_e}{17.38} = \frac{4.19}{3.846} \qquad R_e = 18.93$$

The H plane radius is found from similiar triangles like the E plane.



$$\frac{R_H}{17.63} = \frac{5.69}{5} \qquad R_H = 20.06$$

Suppose we want to analyze this horn at 8 GHz. The wavelength is found from

$$\lambda = \frac{11.80285 \times 10^9 \text{ inches/sec}}{8 \times 10^9 \text{ cycles/sec}} = 1.475 \text{ inches}$$

Using this wavelength, we can find the maximum phase deviation in the E and H planes.

$$S_e = \frac{H^2}{8\lambda R_e} = \frac{(8.38)^2}{8(1.475)18.93} = 0.31$$

$$S_H = \frac{W^2}{8\lambda R_H} = \frac{(11.38)^2}{8(1.475)\,20.06} = 0.55$$

These maximum phase deviations in wavelengths indicates which curves are to be used on the universal radiation pattern curves. We can construct the following table with the E and H plane patterns.

θ	$\frac{W}{\lambda}\,\mathrm{Sin}\,\theta$	Rel. Field Strength	$(1 + \mathrm{Cos}\,\theta)/2$	Pattern
5	.673	.795	.998	-2.
10	1.339	.5	.992	-6.1
15	1.996	.27	.983	-11.5
20	2.64	.15	.970	-16.74

E Plane

θ	$\frac{H}{\lambda}\,\mathrm{Sin}\,\theta$	Rel. Field Strength	$(1 + \mathrm{Cos}\,\theta)/2$	Pattern
5	.495	.703	.998	-3.1
10	.986	.41	.992	-7.8
15	1.47	.36	.983	-9.0
20	1.94	.14	.97	-17.3
25	2.4	.19	.953	-14.9
30	2.84	.085	.933	-22.

We can use the universal radiation patterns to find the 3 dB beamwidths of the horn. The half beamwidth is about 5 degrees, so the $(1 + \mathrm{Cos}\,\theta)/2$ factor is about 0.998. The 3 dB beamwidth is 0.7071 which when divided by the $(1 + \mathrm{Cos}\,\theta)/2$ factor is 0.708. In the H plane the factor W/λ Sin θ equals 0.87; $\theta = \mathrm{Sin}^{-1}(0.87\,\lambda/W) = 8.5$. The half power beamwidth is 17.1°. The E plane factor is 0.49; $\theta = 4.95^\circ$. The half power beamwidth is 9.9°. When we use Kraus's formula for directivity found on page 35, we get an estimate of the directivity (gain) of the horn.

$$\text{Directivity} = \frac{41253}{\theta_E\,\theta_H} = \frac{41253}{(17.1)(9.9)} \longrightarrow 23.9\text{ dB}$$

The directivity of the rectangular horn can be found from the equation on page 216 for an aperture. When the maximum of the pattern occurs at $\theta = 0$, Schelkunoff has solved this integral and finds the directivity.

$$\text{Directivity} = \frac{8\pi R_h R_e}{W\,H}\left((C(u) - C(v))^2 + (S(u) - S(v))^2\right)\left(C^2(z) + S^2(z)\right)$$

where

$$u = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_H}}{W} + \frac{W}{\sqrt{\lambda R_H}}\right)$$

$$v = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_H}}{W} - \frac{W}{\sqrt{W R_H}}\right) \qquad z = \frac{H}{\sqrt{2\lambda R_e}}$$

and

$$\left.\begin{aligned} C(x) &= \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \\ S(x) &= \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \end{aligned}\right\} \quad \text{Fresnel Integrals}$$

If the horn is flared only in the E plane, then the directivity is

$$g_e = \frac{64\,W\,R_e}{\pi\lambda H}\left(C^2(z) + S^2(z)\right)$$

Similiarly, if the horn is flared only in the H plane, the directivity is

$$g_m = \frac{4\pi H\,R_h}{\lambda W}\left((C(u) - C(v))^2 + (S(u) - S(v))^2\right)$$

These functions are plotted on pages 225 through 228. They are the gains for horns flared in one direction only. We can use these to find the gain of a horn flared in both directions.

$$g = \frac{\pi}{32}\left(\frac{g_m\lambda}{H}\right)\left(\frac{g_e\lambda}{W}\right)$$

We will use these curves to find the gain of our example horn. These are the parameters of the horn.

$W = 11.38$ (H plane width)

$H = 8.38$ (E plane height)

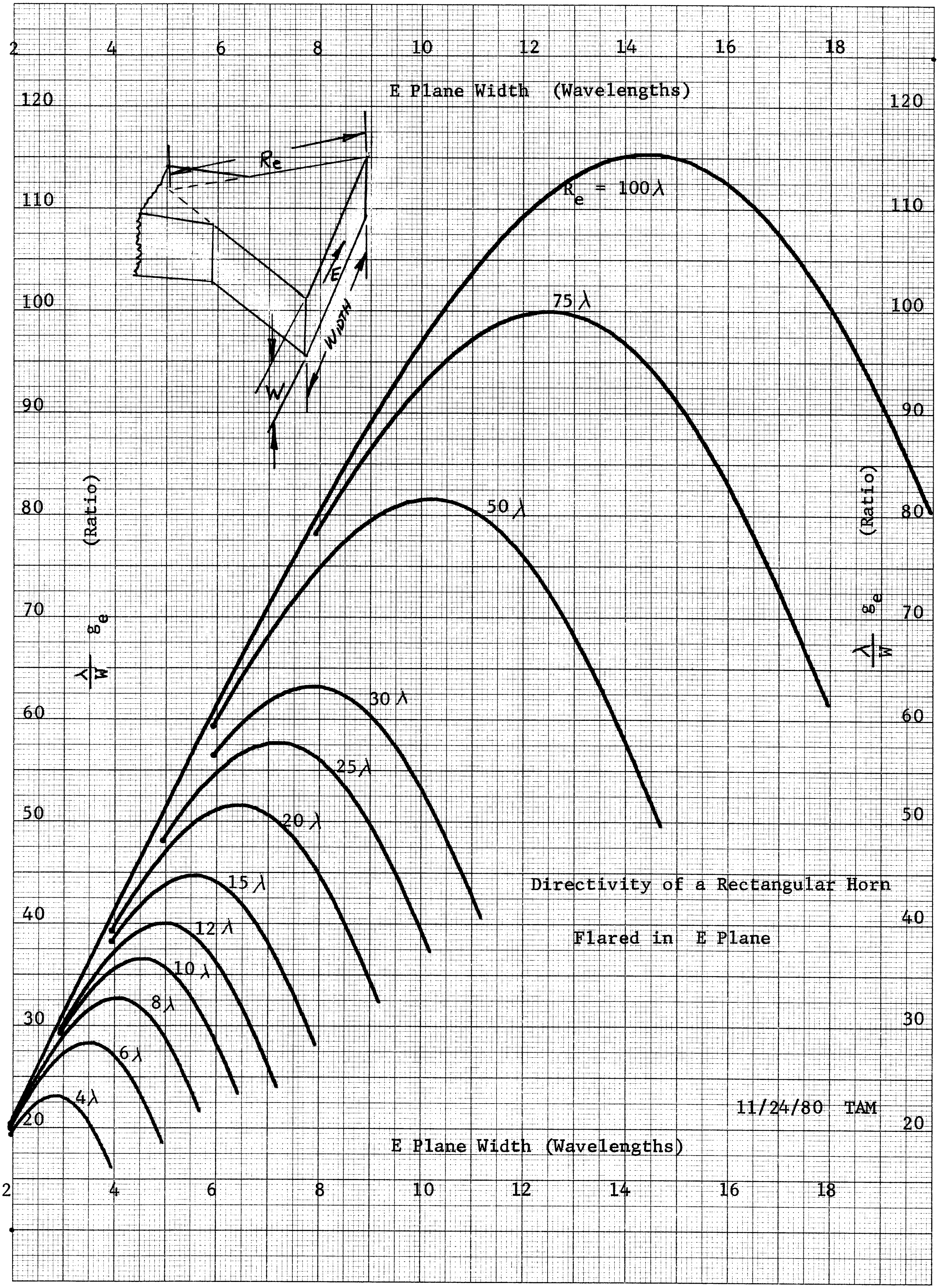
$R_h = 20.06$ (H plane slant length)

$R_e = 18.93$ (E plane slant length)

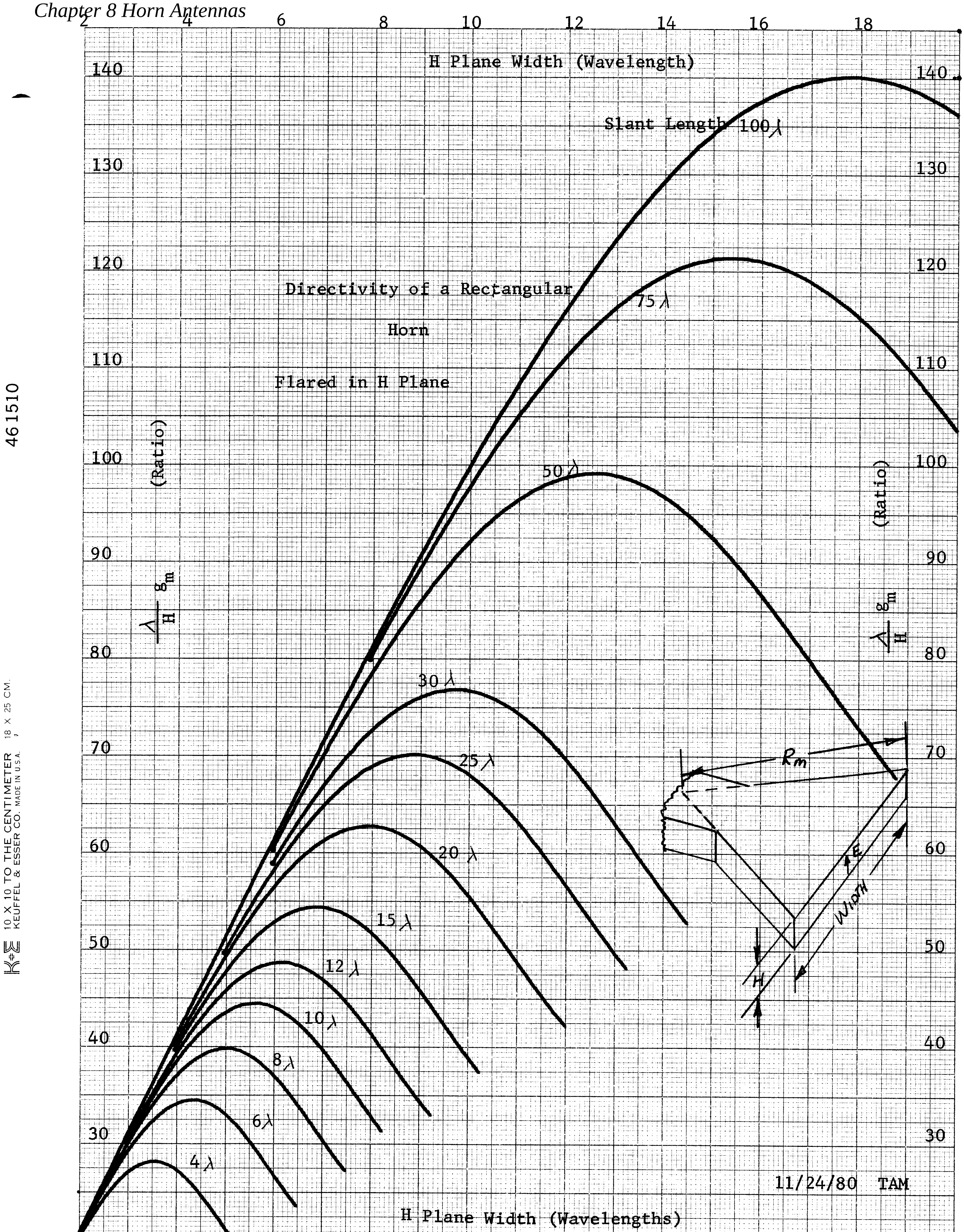
Frequency = 8 GHz $\lambda = 1.475$ inches

$W/\lambda = 7.71$ $R_h/\lambda = 13.6$ $H/\lambda = 5.68$ $R_e/\lambda = 12.8$

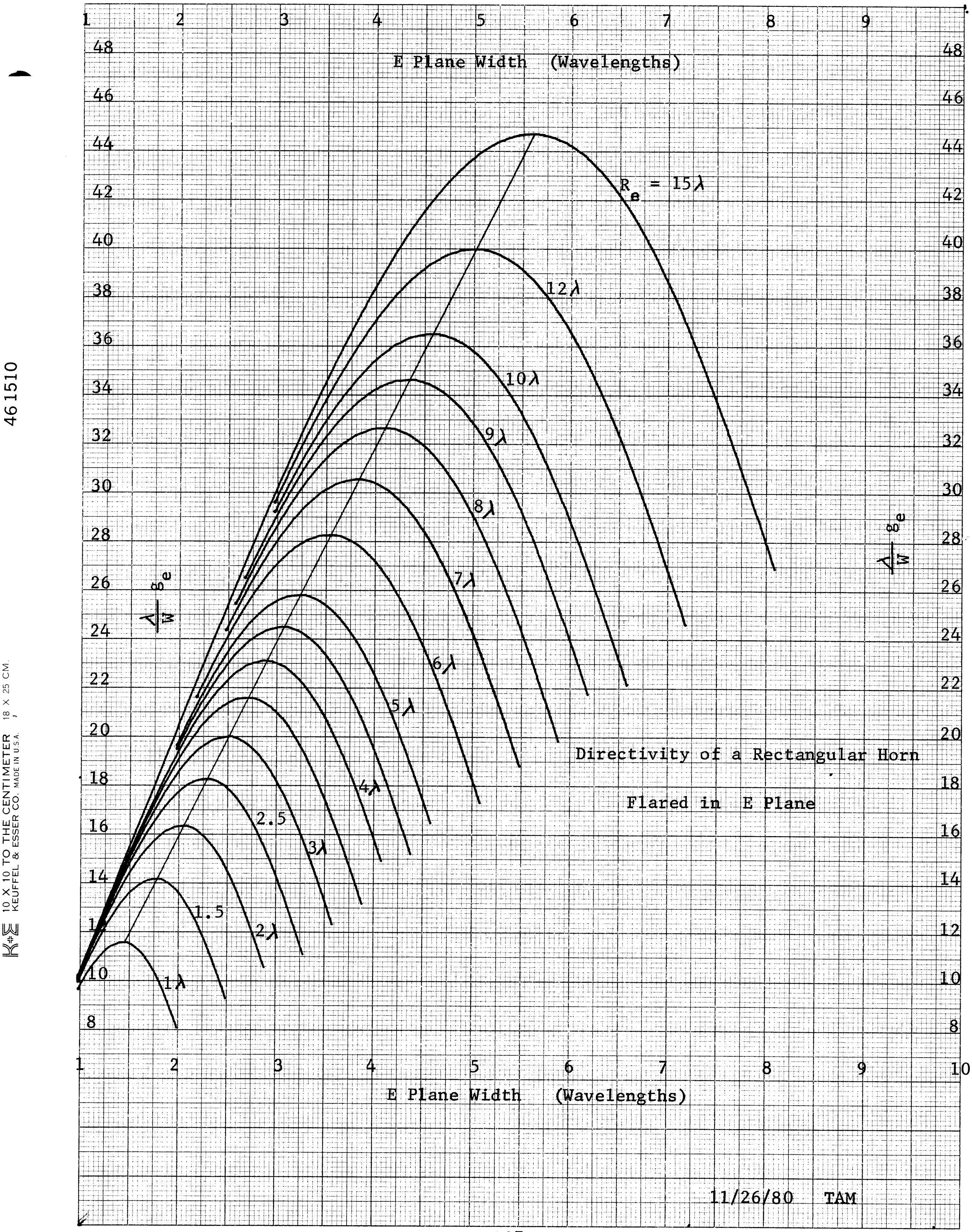
$$\left(\frac{\lambda g_e}{W}\right) = 40.2 \qquad \left(\frac{\lambda g_H}{H}\right) = 48.6$$



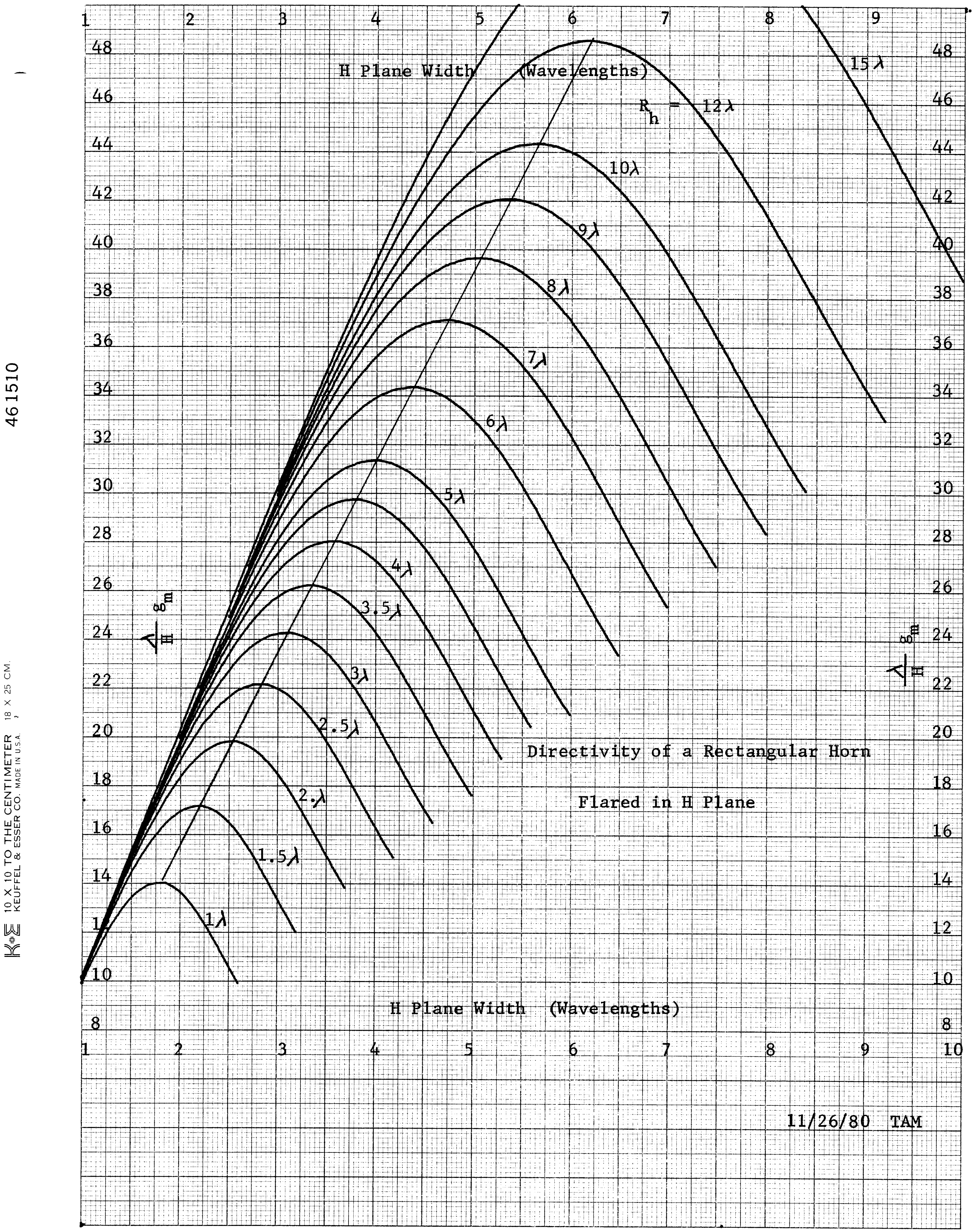
E Plane Width (Wavelengths)
R_e
E
WIDTH
W
$R_e = 100\lambda$
75λ
50λ
30λ
25λ
20λ
15λ
12λ
10λ
8λ
6λ
4λ
(Ratio)
$\frac{W}{\lambda} g_e$
Directivity of a Rectangular Horn
Flared in E Plane
11/24/80 TAM
E Plane Width (Wavelengths)



H Plane Width (Wavelength)
Slant Length 100λ
75λ
50λ
30λ
25λ
20λ
15λ
12λ
10λ
8λ
6λ
4λ
Directivity of a Rectangular
Horn
Flared in H Plane
(Ratio)
$\frac{\lambda}{H}g_m$
R_m
E
WIDTH
H
11/24/80 TAM
H Plane Width (Wavelengths)
2
4
6
8
10
12
14
16
18
30
40
50
60
70
80
90
100
110
120
130
140



E Plane Width (Wavelengths)
$R_e = 15\lambda$
12λ
10λ
9λ
8λ
7λ
6λ
5λ
4λ
3λ
2.5
2λ
1.5
1λ
$\frac{\lambda}{W} G_e$
Directivity of a Rectangular Horn
Flared in E Plane
11/26/80 TAM


H Plane Width (Wavelengths)
R_h = 12λ
15λ
10λ
9λ
8λ
7λ
6λ
5λ
4λ
3.5λ
3λ
2.5λ
2.λ
1.5λ
1λ
$\frac{\lambda}{H} G_m$
Directivity of a Rectangular Horn
Flared in H Plane
H Plane Width (Wavelengths)
11/26/80 TAM
46 1510
K+E 10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.

$$g = \frac{\pi}{32}(40.2)(48.6) = 191.8 \rightarrow 22.8\ dB$$

Notice the difference between this and the value given by Kraus's method using beamwidths. The nomograph on page 39 using the conical beam method gives the gain at 22.8 dB for the beamwidths of 9.9 and 17.1 degrees. The manufacturer also lists the gain as 22.8 dB at this frequency. This horn is used as a gain standard against which other antennas are measured.

OPTIMUM RECTANGULAR HORN

There is one particular horn which has come to be called the optimum horn. It has the following properties: 1) Minimum E plane radius or H plane radius, and 2) equal E and H plane beamwidths. If we take the curve on page 227 and draw a line through the points of maximum gain for a given R_e, then we find that the points of maximum gain for all R_e line up approximately on the same line. At this point the distance R_e will be minimized for a given gain. We can find the maximum aperture phase deviation in wavelengths for this aperture in the E plane.

$$R_e = 12\lambda \qquad H = 5\lambda \qquad S_e = \frac{H^2}{8\ R_e} = \frac{5^2}{8\ (12)} = 0.26$$

In the H plane the maximum aperture phase deviation in wavelengths is found from the similiar line drawn on the curve on page 228.

$$R_h = 8\lambda \qquad W = 5.05\lambda \qquad S_h = \frac{W^2}{8\ R_h} = \frac{(5.05)^2}{8\ (8)} = 0.40$$

We can use the universal radiation patterns to find the ratio of the height to the width which gives equal E and H plane beamwidths. If we use these values of S, then the abscissas at 0.707 ordinate are:

$$\frac{H}{\lambda} \sin\theta = 0.475 \qquad \frac{W}{\lambda} \sin\theta = 0.68$$

$$\frac{H}{W} = 0.7$$

The gain for an optimum sectoral horn is a linear function of the width or height of the horn. If we pick the ratio of the height to the width of the horn to be 0.7, we can find the gain and multiplication factors for the design. Pick $W = 5\lambda$.

$$\frac{\lambda}{H} g_h = 39.3 \quad \text{on optimum line}$$

The height will be $(0.7)\ 5 = 3.5$ to give an equal beamwidth; the gain at the optimum line in the E plane is given on the graph.

$$\frac{\lambda}{W} g_e = 28$$

The gain of the horn is: $$\frac{\pi}{32}\left(\frac{\lambda}{W} g_e\right)\left(\frac{\lambda}{H} g_h\right) = 108$$

Since the gain is found as the product of two factors, each width will be a linear function of the square root of the gain.

$$K_w \sqrt{108} = 5 \qquad K_w = 0.481$$

$$K_h \sqrt{108} = 3.5 \qquad K_h = 0.337$$

The optimum horn may be found from these equations:

$$W/\lambda = 0.481 \sqrt{G}$$

$$H/\lambda = 0.337 \sqrt{G}$$

The slant lengths are found as $R_h/\lambda = \dfrac{(W/\lambda)^2}{8\ S_h \lambda}$ & $R_e/\lambda = \dfrac{(H/\lambda)^2}{8\ S_e \lambda}$

$$R_h/\lambda = \frac{(0.481 \sqrt{G})^2}{8\ (0.4)} = 0.0723\ G$$

$$R_e/\lambda = \frac{(0.337 \sqrt{G})^2}{8\ (0.26)} = 0.0546\ G$$

Where G is the power gain as a ratio: $G = 10^{(G\ (dB)/10)}$

Suppose we want to design a horn with 22 dB gain.

$$G = 10^{(22/10)} = 158.5$$

$$W/\lambda = 0.481 \sqrt{158.5} = 6.055$$

$$H/\lambda = 0.337 \sqrt{158.5} = 4.242$$

$$R_e/\lambda = 0.0546\ (158.5) = 8.654$$

$$R_h/\lambda = 0.0723\ (158.5) = 11.460$$

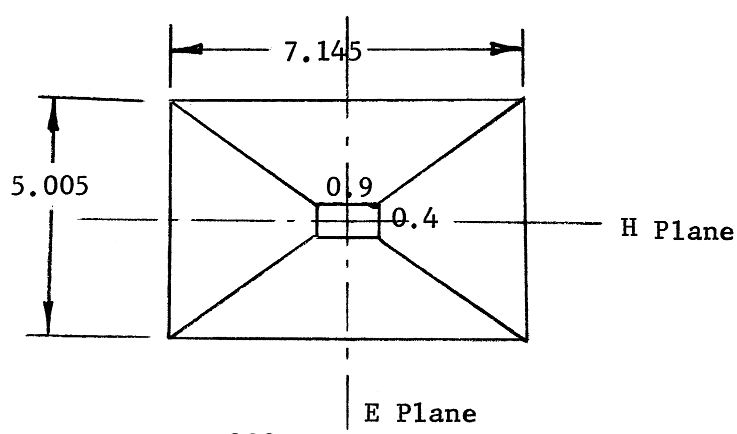
The center frequency of the horn will be at 10 GHz and the horn will be fed from WR-90 waveguide which has dimensions 0.900 x 0.400 inches.

$$\lambda = 1.18 \text{ inches}$$

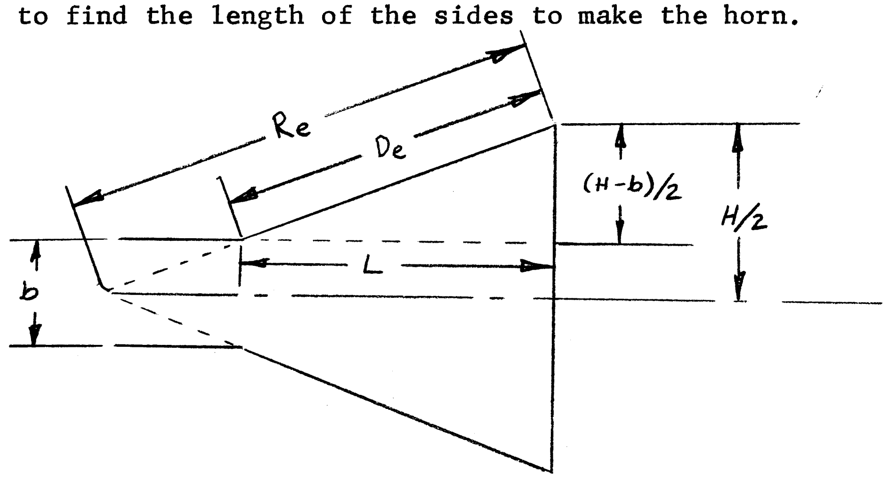
$$W = 7.145 \qquad H = 5.005$$

$$R_h = 13.523 \qquad R_e = 10.212$$

Looking at the mouth of the horn, it looks like the figure below.



We need to find the length of the sides to make the horn.



Above is a cross sectional view through the E plane of the horn.

$$D_e^2 = L^2 + (H - b)^2/4$$

From similiar triangles we find:

$$\frac{R_e}{D_e} = \frac{H}{H - b}$$

We can substitute this expression in the equation above and solve for the axial length of the horn in the E plane.

$$R_e^2 \left(\frac{H-b}{b}\right)^2 = L_e^2 + (H-b)^2/4$$

$$L_e = \frac{H-b}{H}\sqrt{R_e^2 - H^2/4}$$

Similiarly, $$L_H = \frac{W-a}{W}\sqrt{R_H^2 - W^2/4}$$

We can use these equations to find the axial lengths in both planes.

$$L_e = \frac{5.005 - .4}{5.005}\sqrt{(10.212)^2 - (5.005)^2/4} = 9.109$$

$$L_H = \frac{7.145 - .9}{7.145}\sqrt{(13.523)^2 - (7.145)^2/4} = 11.400$$

Because the lengths are different, we will have a problem building the horn. The H plane must start flaring before the E plane. It would be nice to restrict the horn design to have equal axial lengths so that the horn will join the waveguide at a single plane. We can solve for the slant radius in the H plane in terms of the axial length.

$$R_H = \frac{W}{W-a}\sqrt{L^2 + (W-a)^2/4}$$

With axial length

$$L = \frac{H-b}{H}\sqrt{R_e^2 - H^2/4}$$

If we solve for R_h, then

$$R_H = \frac{7.145}{7.145-.9}\sqrt{(9.109)^2 + (7.145-0.9)^2/4} = 11.017$$

Since we did not use the optimum design, the gain of the horn may not be the required gain. We will need to calculate the gain of this horn. The E plane gain has not changed when we change the H plane slant radius.

$$H/\lambda = 4.242 \qquad R_e/\lambda = 8.654$$

From the plot on page 227, we find the gain factor for the E plane.

$$\frac{\lambda}{W} g_e = 33.9$$

In the H plane we use the plot on page 228.

$$W/\lambda = 6.055 \qquad R_h/\lambda = \frac{11.017}{1.18} = 9.336$$

This point is to the right of the optimum line for an H plane sectorial horn.

$$\frac{\lambda}{H} g_h = 41.8$$

The gain of the horn is $\frac{\pi}{32}\left(\frac{\lambda}{W} g_e\right)\left(\frac{\lambda}{H} g_h\right) = 139.1$

The gain has fallen below the required 158.5 (22 dB). We need to design a horn with higher gain so that when we apply the restriction on the H plane slant radius to give equal E and H plane axial lengths, the gain will be 22 dB. We will increase the gain by the ratio of the last designed gain to the present gain times the required gain.

$$g = g_r\, g_d / g_a$$

Where g is the new design gain, g_r is the required gain, g_d is the last design gain, and g_a is the actual gain of the present design. We are setting up an iteration process. The new design gain is

$$g = (158.5)(158.5)/139.1 = 181.5$$

Using this design gain, we can find the dimensions of the horn.

$$W/\lambda = .481\sqrt{181.5} = 6.480 \qquad W = 7.646 \text{ inches}$$

$$H/\lambda = .337\sqrt{181.5} = 4.540 \qquad H = 5.357$$

$$R_e/\lambda = .0546\,(181.5) = 9.910 \qquad R_e = 11.694$$

We find the axial length using these dimensions and the H plane slant radius.

$$L = \frac{H-b}{H}\sqrt{R_e^2 - H^2/4} = 10.533 \text{ IN.}$$

$$R_H = \frac{W}{W-a}\sqrt{L^2 + (W-a)^2/4} = 12.535$$

$$R_H/\lambda = 10.623$$

We can find the gain of this horn from the plots on page 227 and 228.

$$\frac{\lambda}{W} g_e = 36.3 \qquad \frac{\lambda}{H} g_h = 44.4$$

$$g_a = 158.2$$

This is quite close to the required gain of 158.5 (less than 0.01 dB) and we have our final design.

$W = 7.646 \quad H = 5.357 \quad R_h = 12.535 \quad R_e = 11.694$

Axial length = 10.533

Let us check how closely the antenna comes to having equal beamwidths.

$$S_e = \frac{H^2}{8R_e\lambda} = .26 \qquad S_h = \frac{W^2}{8R_h\lambda} = .49$$

$$\frac{H}{\lambda}\sin\theta_e = .475 \qquad \frac{W}{\lambda}\sin\theta_h = .775$$

$$2\theta_E = 12^\circ \qquad 2\theta_H = 13.7^\circ$$

Because we forced the horn to join the waveguide in a single plane we have removed the restriction that the antenna have equal 3 dB beamwidths, although they are still close.

Let us try keeping the axial length of the H plane and forcing the E plane slant radius to give this length. This will give us another design.

$$R_e = \frac{H}{H-b}\sqrt{L^2 + (H-b)^2/4}$$

$$R_e = \frac{5.005}{5.005-.4}\sqrt{(11.4)^2 + (5.005-.4)^2/4} = 12.64 \text{ INCHES}$$

We can find the gain of this horn using the plots on pages 227 and 228.

$$W/\lambda = 6.055 \qquad H/\lambda = 4.242$$

$$R_H/\lambda = 11.460 \qquad R_e/\lambda = 10.71$$

$$\left(\frac{\lambda}{H} g_h\right) = 47.4 \qquad \left(\frac{\lambda}{W} g_e\right) = 37$$

$$g_a = 172.2$$

The new design gain is $$g = \frac{(158.5)^2}{172.2} = 145.9$$

Now we can design a horn with this gain.

$$W/\lambda = .481\sqrt{145.9} = 5.81 \qquad W = 6.856$$

$$H/\lambda = .337\sqrt{145.9} = 4.070 \qquad H = 4.803$$

$$R_H/\lambda = .0723\,(145.9) = 10.548 \qquad R_H = 12.447$$

$$L = \frac{W-a}{W}\sqrt{R_H^2 - W^2/4} = 10.395$$

$$R_e = \frac{H}{H-b}\sqrt{L^2 + (H-b)^2/4} = 11.591$$

We can find the gain of this horn.

$$R_H/\lambda = 10.548 \qquad W/\lambda = 5.81 \qquad \left(\frac{\lambda}{H} g_h\right) = 45.6 \qquad R_e/\lambda = 9.822 \qquad H/\lambda = 4.070 \qquad \left(\frac{\lambda}{W} g_e\right) = 35.4$$

$$g = \frac{\pi}{32}\left(\frac{\lambda}{H} g_h\right)\left(\frac{\lambda}{W} g_e\right) = 158.48$$

This horn also has the required gain. We can calculate the beamwidths of the antenna.

$$S_e = \frac{H^2}{8R_e\lambda} = .21 \qquad S_h = \frac{W^2}{8R_h\lambda} = .40$$

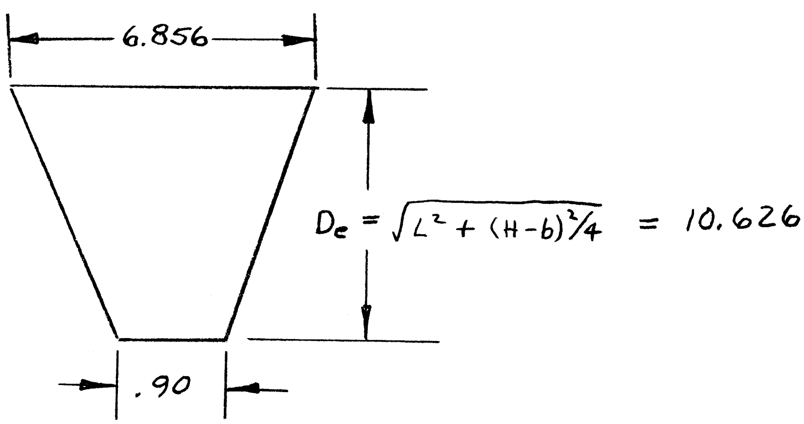
$$\frac{H}{\lambda}\sin\theta_E = .46 \qquad \frac{W}{\lambda}\sin\theta_h = .69$$

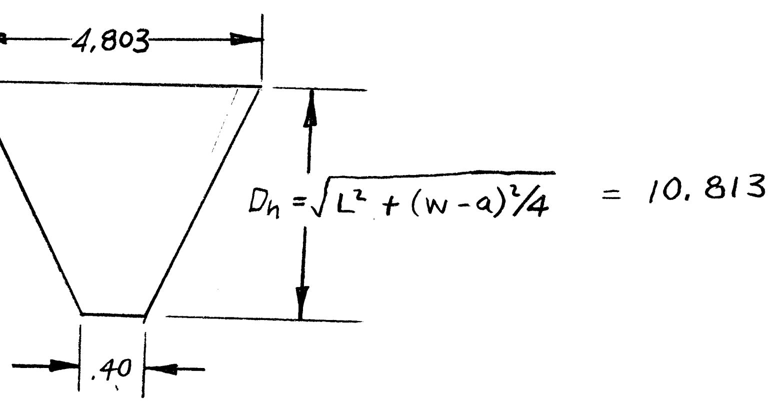
$$2\theta_E = 13^\circ \qquad 2\theta_h = 13.6^\circ$$

Let us compare the two designs.

		Design 1	Design 2
Width	W	7.646	6.856
Height	H	5.357	4.803
Axial Length	L	10.623	10.395
E Plane Slant	R_e	11.694	11.591
H Plane Slant	R_h	12.535	12.447
E Plane Beamwidth		12	13
H Plane Beamwidth		13.7	13.6

It appears that the second design is more optimum. The dimensions of the side plates are:





These dimensions do not allow for the overlap needed to solder the sides of the horn together.

It is possible to design rectangular horns with different beamwidths in the E and H planes. Since there are extra parameters available, this horn has great possibilities to satisfy a variety of requirements.

For an example let us design a horn with an 80^{o} ten dB beamwidth. The obliquity factor $(1 + \cos\theta)/2 = 0.883$. The 10 dB level on the universal radiation pattern is 0.358 (0.316/0.883). Let us use $S_h = 0.40$.

$$W/\lambda \ \sin\theta = 1.35 \qquad W/\lambda = 1.35/\sin\theta = 2.1$$

$$R_h = (W/\lambda)^2/(8\ S_h) = 1.378$$

Let us pick a frequency of 10 GHz and WR-90 waveguide again. $\lambda = 1.18$ in.

$$W = 2.478 \qquad R_h = 1.626 \qquad L = \frac{W - a}{W}\sqrt{R_h^2 - W^2/4} = .670$$

Assume that $S_e = 0.2$ which we can verify when the design is complete.

$$H/\lambda = .775/\sin\theta = 1.206 \qquad H = 1.423$$

$$R_e = \frac{H}{H - b}\sqrt{L^2 + (H-b)^2/4} \qquad R_e = 1.172$$

Now we can check the value of S_e

$$S_e = \frac{(1.423)^2}{8\ \lambda\ 1.172} = .183$$

Which is close to the assumed value.

The final design is:

$W = 2.478 \qquad H = 1.423 \qquad R_h = 1.626 \qquad R_e = 1.172$

Axial Length = .670

TE and TM Modes

On page 203 and following, we were able to find a solution for the dominant mode waveguide fields by considering two waves traveling at an angle to a reflecting boundary and satisfying the boundary conditions. The electric field was parallel to the plane of reflection. The solution that was found is called transverse electric or the TE mode solution. There is no electric field in the direction of propagation of the wave, but there is a magnetic field in the direction of propagation. In general, any field problem can be solved as a sum of TE and TM (Transverse Magnetic) modes. We will use the vector potentials to derive TE and TM mode solutions.

Let us take Z as the direction of propagation of the mode. We can derive a TE mode solution using the following electric vector potential.

$$\bar{F} = \bar{a}_z \psi$$

Where ψ is a scalar function. The fields can be found from the potential by:

$$\bar{E} = -\nabla \times \bar{F} \qquad \bar{H} = -j\omega\epsilon\bar{F} + \frac{\nabla(\nabla\cdot\bar{F})}{j\omega\mu}$$

Since $\bar{E}$ is a curl of the vector $\bar{F}$ and $\bar{F}$ is only in the Z direction, the electric field is orthogonal to $\bar{F}$ and the direction of propagation. In rectangular coordinates the above equations for the fields expand to:

$$E_x = -\frac{\partial\psi}{\partial y} \qquad H_x = \frac{1}{j\omega\mu}\frac{\partial^2\psi}{\partial x\,\partial z} \qquad \beta = \frac{2\pi}{\lambda}$$

$$E_y = \frac{\partial\psi}{\partial x} \qquad H_y = \frac{1}{j\omega\mu}\frac{\partial^2\psi}{\partial y\,\partial z} \qquad \text{TE (H Mode)}$$

$$E_z = 0 \qquad H_z = \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)\psi$$

These are also called H modes because the magnetic field is in the direction of propagation.

We can find a TM mode solution by using a magnetic vector potential which is in the direction of propagation.

$$\bar{A} = \bar{a}_z \psi$$

Where ψ is a scalar function. The fields from this potential are found from:

$$\bar{H} = \nabla \times \bar{A} \qquad \bar{E} = -j\omega\mu\bar{A} + \frac{\nabla(\nabla\cdot\bar{A})}{j\omega\epsilon}$$

The magnetic field will be zero in the direction of the magnetic potential which is in the direction of propagation. The TM mode can be expanded in rectangular coordinates to give.

$$E_x = \frac{1}{j\omega\epsilon}\frac{\partial^2\psi}{\partial x\partial z} \qquad H_x = \frac{\partial\psi}{\partial y} \qquad \beta = \frac{2\pi}{\lambda}$$

$$E_y = \frac{1}{j\omega\epsilon}\frac{\partial^2\psi}{\partial y\partial z} \qquad H_y = -\frac{\partial\psi}{\partial x} \qquad \text{TM (E Mode)}$$

$$E_z = \frac{1}{j\omega\epsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)\psi \qquad H_z = 0$$

This mode is called the E mode because the electric field is in the direction of propagation.

We will be seeking solutions in source free regions. From pages 121 and 174 we find the differential equations of the vector potentials in the source free regions.

$$\nabla^2\bar{A} + \beta^2\bar{A} = 0$$

$$\nabla^2\bar{F} + \beta^2\bar{F} = 0$$

Since $\bar{A} = \bar{a}_z\psi$ and $\bar{F} = \bar{a}_z\psi$, the vector differential equation reduces to

$$\nabla^2\psi + \beta^2\psi = 0$$

This is called the Helmholtz equation. Note at this point that the scalar function, ψ , is not necessarily the same function for H modes (TE) and E modes (TM). The same partial differential equation is for both TE and TM modes. We will find solution sets for this equation and apply the boundary conditions to get the solutions for each type of mode. In rectangular coordinates the Helmholtz equation is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \beta^2\psi = 0$$

We will use the method of separation of variables with a solution of the form

$$\psi = X(x)\ Y(y)\ Z(z)$$

where X(x), Y(y), and Z(z) are functions of the coordinates x, y, z only. If we substitute this into the equation and divide by ψ , then the equation becomes

$$\frac{1}{X}\frac{d^2X(x)}{dx^2} + \frac{1}{Y}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z}\frac{d^2Z(z)}{dz^2} + \beta^2 = 0$$

Since each term is a function of only one coordinate, then each term is equal to a constant. These constants are called the separation constants. This reduces the problem to three ordinary differential equations.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\beta_x^2 \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\beta_y^2 \qquad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\beta_z^2$$

Where β_x, β_y, and β_z are the separation constants. From the original Helmholtz equation, we obtain an equation between the constants.

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$$

The three equations are identical in form.

$$\frac{d^2X(x)}{dx^2} + \beta_x^2 X(x) = 0$$

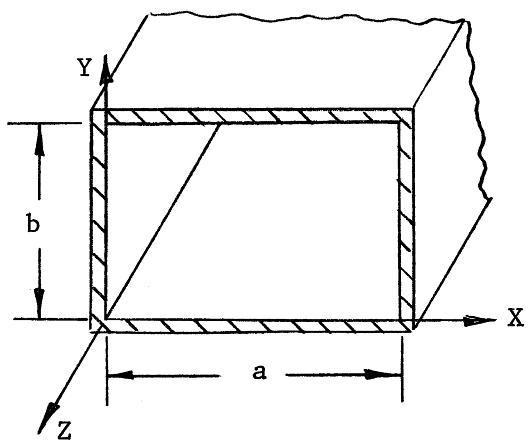
The solution to this equation is called a harmonic function which is denoted $h(\beta_x x)$. The set of harmonic functions is

$$\left\{ \sin\beta_x x,\ \cos\beta_x x,\ e^{j\beta_x x},\ e^{-j\beta_x x} \right\}$$

The trigonometric functions are standing waves and the exponential functions are traveling waves. The solution of the Helmholtz equation is

$$\psi = h(\beta_x x)\, h(\beta_y y)\, h(\beta_z z)$$

RECTANGULAR WAVEGUIDE



We will use the Helmholtz equation solution to find all possible modes of the rectangular waveguide. We will need traveling waves in the Z direction and standing waves in the X and Y directions.

$$\psi = \begin{Bmatrix} \cos\beta_x x \\ \sin\beta_x x \end{Bmatrix} \begin{Bmatrix} \cos\beta_y y \\ \sin\beta_y y \end{Bmatrix} \begin{Bmatrix} e^{-j\beta_z z} \\ e^{j\beta_z z} \end{Bmatrix}$$

The solution is any linear combination of the functions in each column. Each second order differential equation has two arbitrary constants which are determined by the boundary conditions.

The TE modes must have zero tangential electric fields at the walls.

$$E_x = -\frac{\partial \psi}{\partial y} = -\beta_y \begin{Bmatrix} \cos \beta_x x \\ \sin \beta_x x \end{Bmatrix} \begin{Bmatrix} -\sin \beta_y y \\ \cos \beta_y y \end{Bmatrix} e^{-j\beta_z z}$$

At $y = 0$, $E_x = 0$ which is satisfied if $\psi \propto \cos \beta_y y$

At $y = b$ $E_x = 0$ which is satisfied if $\beta_y = \frac{n\pi}{b}$

Where n is an integer. The tangential electric field at $x = 0$ and $x = a$ is

$$E_y = \frac{\partial \psi}{\partial x} = \beta_x \begin{Bmatrix} -\sin \beta_x x \\ \cos \beta_x x \end{Bmatrix} \cos \frac{n\pi}{b} y \; e^{-j\beta_z z}$$

The zero tangential fields are satisfied if $\psi \propto \cos \beta_x x$ and $\beta_x = \frac{m\pi}{a}$ where m is an integer. The electric vector potential in the Z direction (TE mode) becomes

$$\psi_{mn}^{TE} = \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \; e^{-j\beta_z z}$$

from which the TE mode field solution is found using the equations above.

The TM mode potential function is found by satisfying the boundary conditions. The electric field in the Z direction is found from

$$E_z = \frac{1}{j\omega\epsilon} (\beta^2 - \beta_z^2) \psi^{TM}$$

For this to be zero at the walls the following conditions must hold.

$\psi^{TM} = 0$ at $x = 0$ and $y = 0$. This is true if

$$\psi^{TM} = \sin \beta_x x \sin \beta_y y \; e^{-j\beta_z z}$$

For $\psi^{TM} = 0$ at $x = a$ and $y = b$, then $\beta_x = \frac{m\pi}{a}$ and $\beta_y = \frac{n\pi}{b}$ where m and n are integers. The magnetic vector potential for TM mode solutions is

$$\psi_{mn}^{TM} = \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \; e^{-j\beta_z z}$$

As an exercise it should be shown that this potential also satisfies the boundary conditions on E_x and E_y at the walls.

Both the TE and TM modes satisfy the same equation for the propagation constant

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$$

$$\beta_z^2 = \beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

The mode can propagate if the wavelength is less than the cut off wavelength for that mode in the waveguide. The cut off wavelength is found from

$$\beta_c^2 = \left(\frac{2\pi}{\lambda_c}\right)^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The waveguide wavelength is found by solving the propagation equation for wavelength in the Z direction.

$$\lambda_g = \frac{\lambda_0}{\sqrt{1 - (\lambda_0/\lambda_c)^2}} \qquad (\lambda_c)_{mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

λ_0 = Free Space Wavelength

Note that the TE_{mn} and TM_{mn} modes have the same cut off frequencies for a given m and n.

The fields of the TE modes in rectangular waveguide are found from the electric vector potential in the Z direction.

TE Mode

$$E_x = \frac{n\pi}{b} E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$E_y = -\frac{m\pi E_0}{a} \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$H_x = \frac{\beta_z E_0}{\omega\mu} \frac{m\pi}{a} \sin\left(\frac{m\pi}{x}a\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$H_y = \frac{\beta_z E_0}{\omega\mu} \frac{n\pi}{b} \cos\left(\frac{m\pi}{x}a\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$H_z = \frac{E_0}{j\omega\mu} \left(\beta^2 - \beta_z^2\right) \cos\left(\frac{m\pi}{x}a\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

These equations are for a wave traveling in the positive Z direction.

Similiarly we find the TM mode fields in the rectangular waveguide from the magnetic vector potential in the Z direction.

TM Mode

$$E_x = \frac{m\pi}{a} E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$E_y = \frac{n\pi}{b} E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$E_z = \frac{E_0}{j\omega\epsilon} \left(\beta^2 - \beta_z^2\right) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$H_x = \frac{n\pi}{b} E_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

$$H_y = -\frac{m\pi}{a} E_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-j\beta_z z}$$

We can use these equations to find the Z directed impedance of the wave in the waveguide.

$$(Z_o)_{mn} = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

$$(Z_o)_{mn}^{TE} = \frac{\omega\mu}{\beta_z} = \frac{\omega\mu}{\beta\sqrt{1-(f_c/f)^2}}$$

$$(Z_o)_{mn}^{TM} = \frac{\beta_z}{\omega\epsilon} = \frac{\beta\sqrt{1-(f_c/f)^2}}{\omega\epsilon}$$

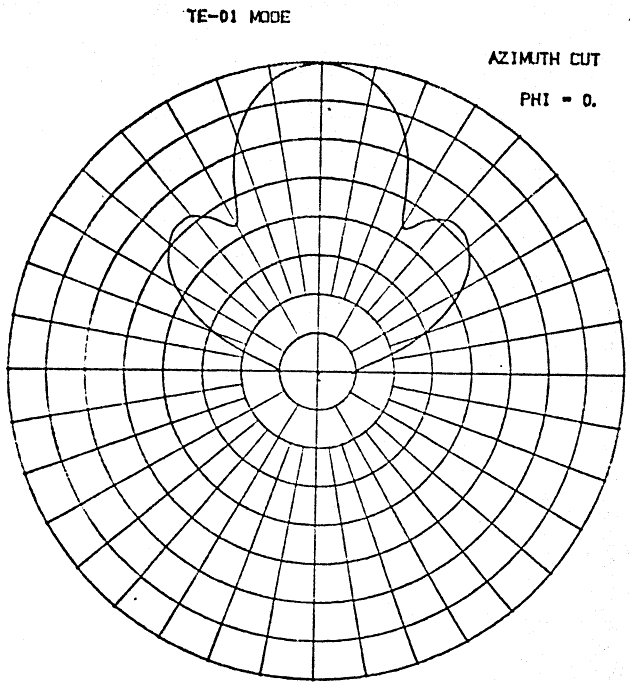
Substituting $\beta = \omega\sqrt{\mu\epsilon}$, the Z directed wave impedances become

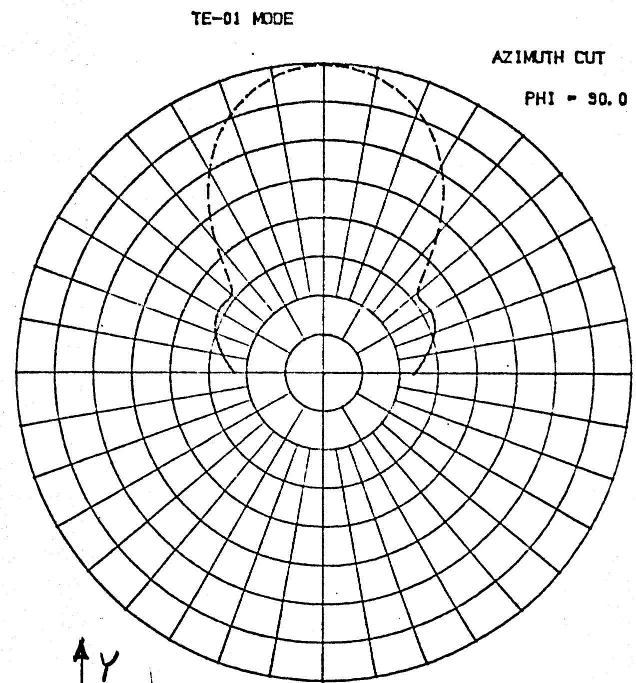
$$Z_{o\,mn}^{TE} = \frac{\eta}{\sqrt{1-(f_c/f)^2}} \qquad Z_{o\,mn}^{TM} = \eta\sqrt{1-(f_c/f)^2}$$

As the cut off frequency is approached, the TE mode wave impedance approaches infinity and the TM mode wave impedance approaches zero. For both cases for frequencies much greater than the cut off frequency, the wave impedance approaches that of free space. In terms of waves reflecting off the walls we find that the waves traveling in the waveguide reflect off the walls at greater distances. Below cut off the TE mode wave impedance is inductive and the TM mode impedance is capacitive. Obstacles in waveguides are either inductive or capacitive depending on whether higher order TE modes or TM modes which are generated to satisfy the boundary conditions have the most energy.

We can use the equations of the electric fields to find the patterns of a horn excited in these modes by using the Huygens source approximation. On pages 243 through 248 are patterns of 2 wavelength square aperture horns excited in the first few rectangular modes. When designing horns, the higher order modes can be generated whenever the dimensions will support them and obstacles excite them. The extra modes can be detected from the patterns. Of course, real horn patterns will be a combination of modes.

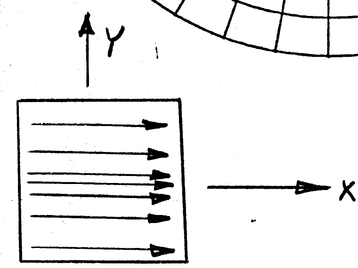
On page 249 and 250 are the E and H plane patterns of a horn with two modes, TE_{01} and TE_{02}. This horn has the same dimensions as the horn patterns on pages 243 and 244. Notice that the pattern electric fields add. At boresight the pattern is just the TE_{01} mode. The TE_{02} mode changes sign by 180 degrees as the pattern passes through a null at bore-sight. The two patterns add on the left side and substract on the right

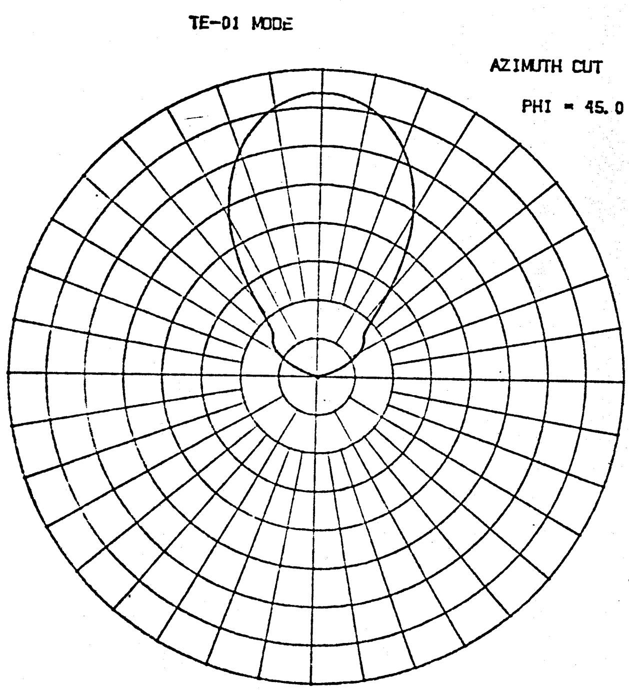


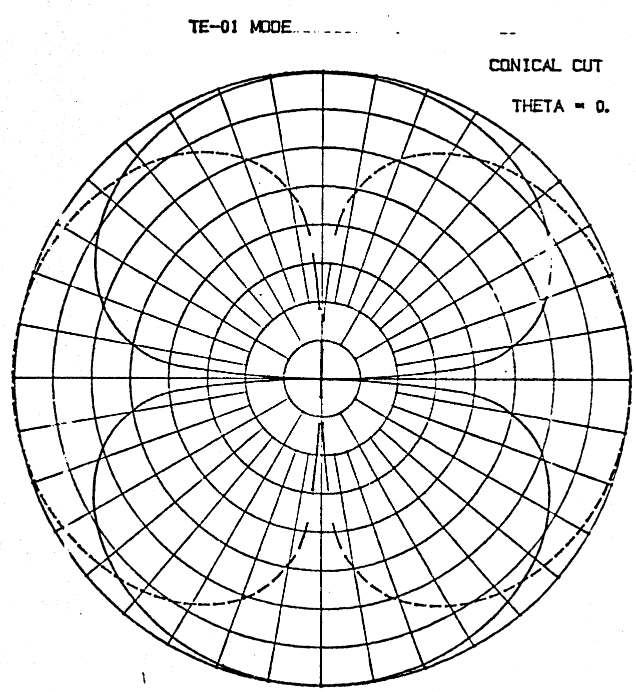


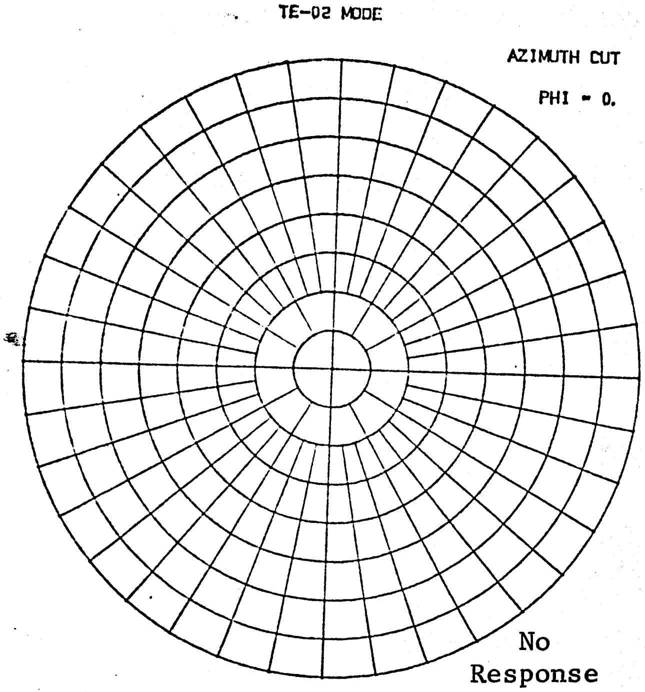
Solid - Theta Component

Dash - Phi Component

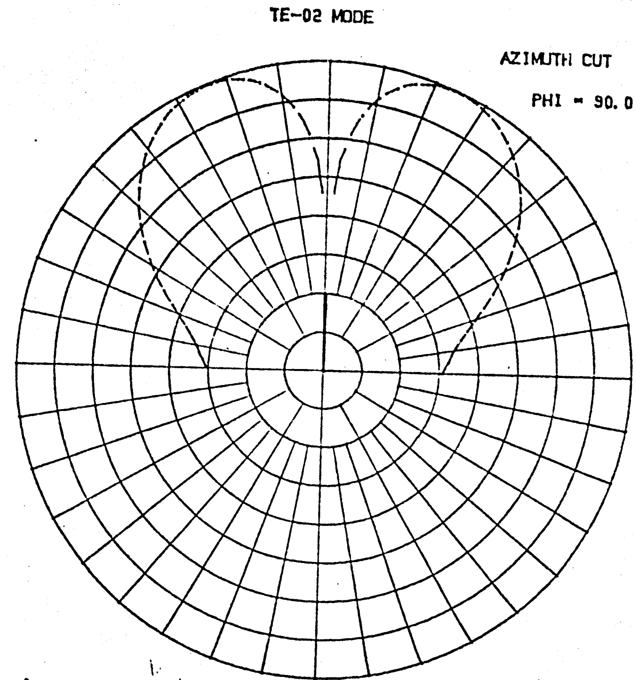




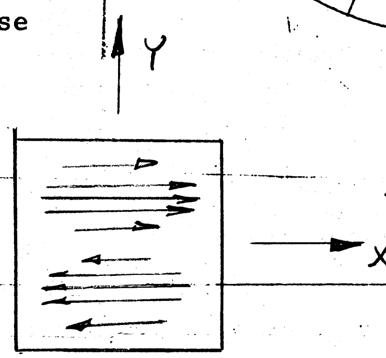


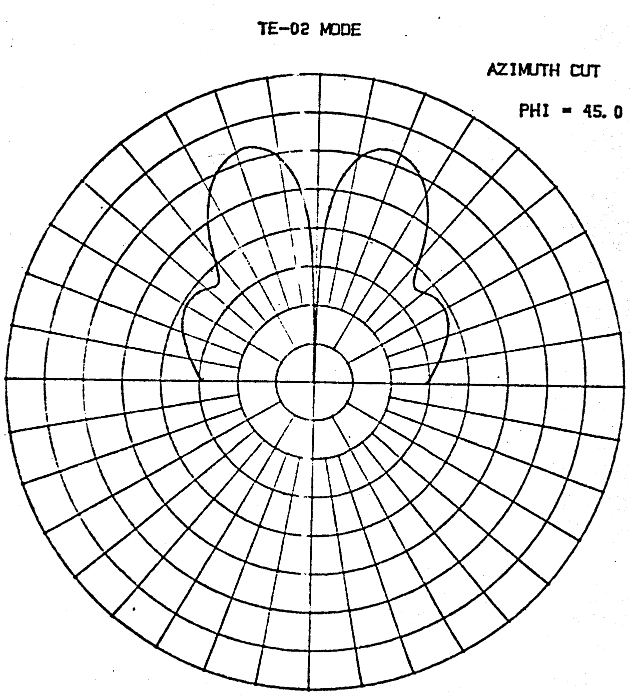


No
Response

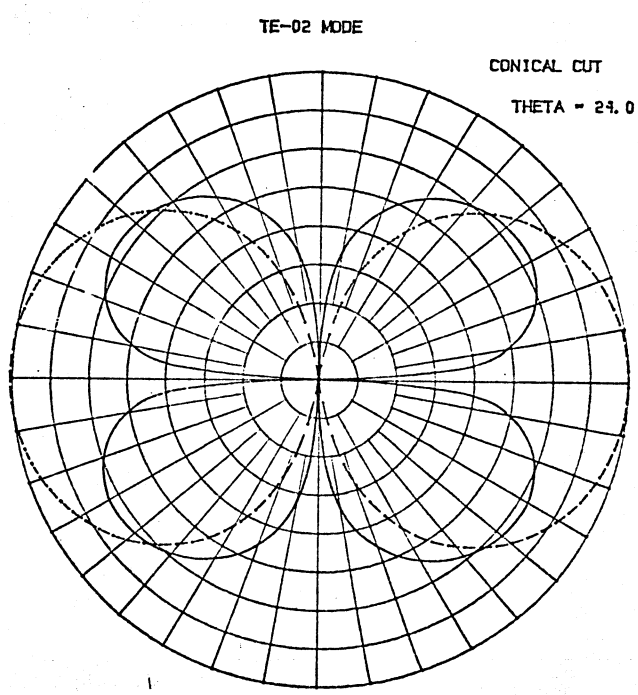


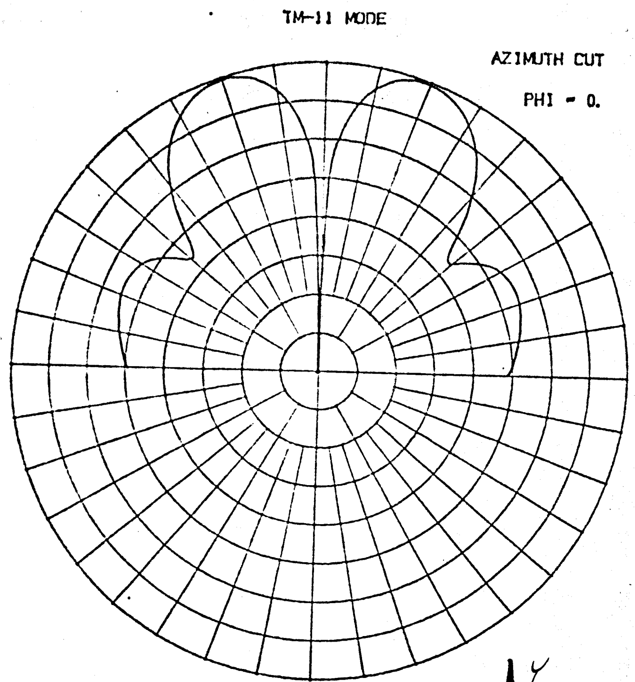
Solid - Theta Component
Dash - Phi Component

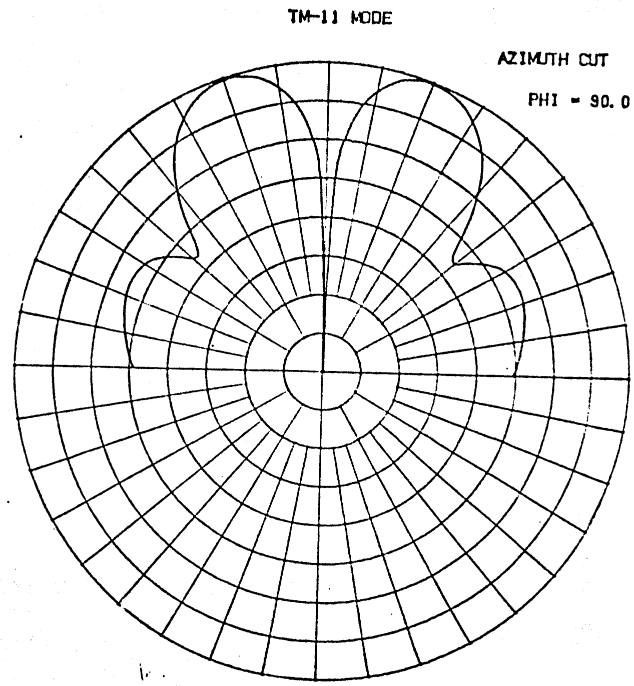


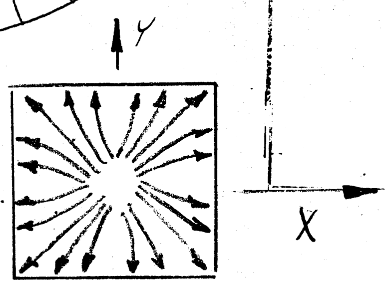


Phi Response is the same as
Theta Response.

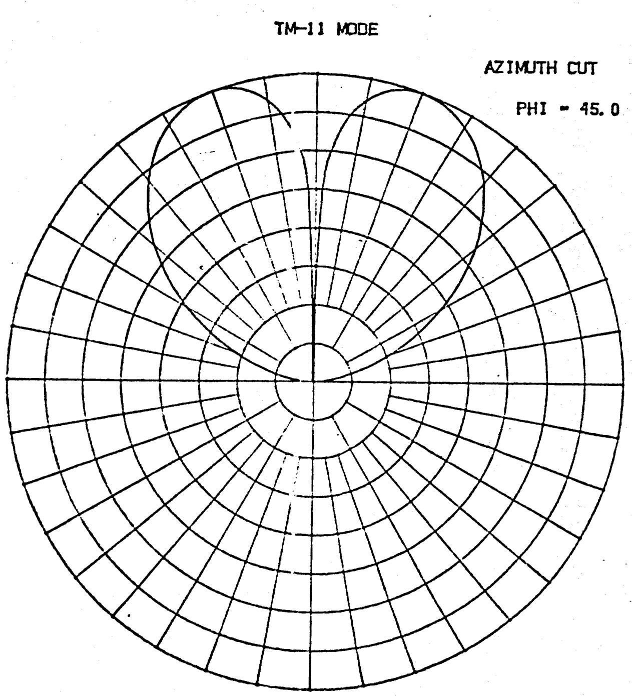


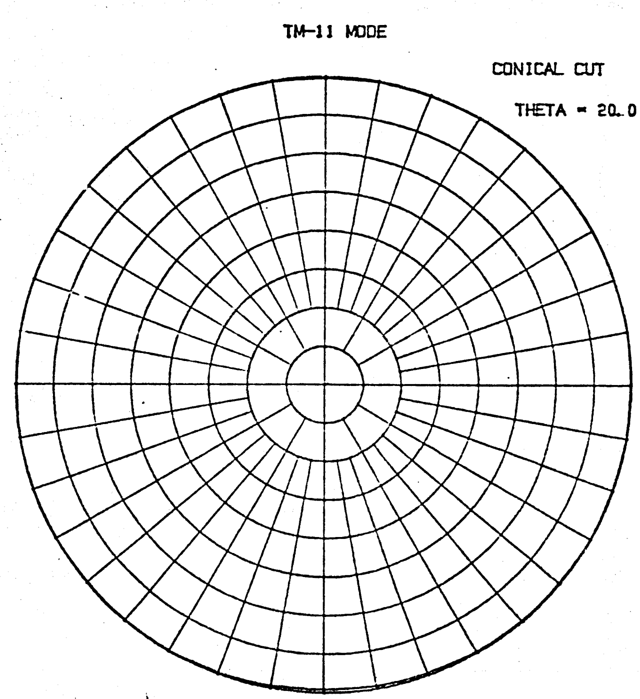




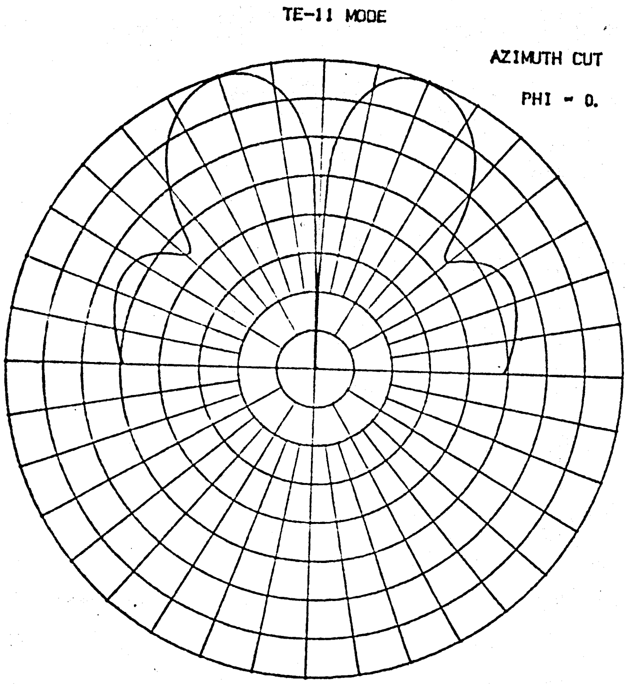


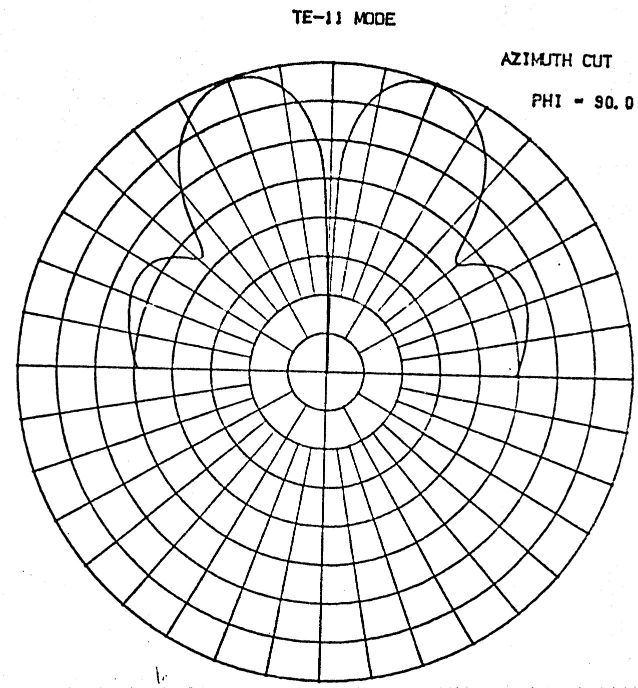
Solid - Theta Component

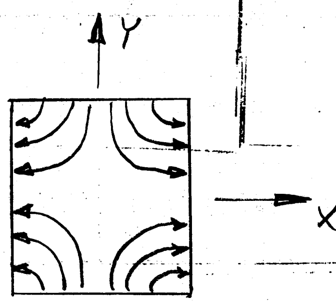




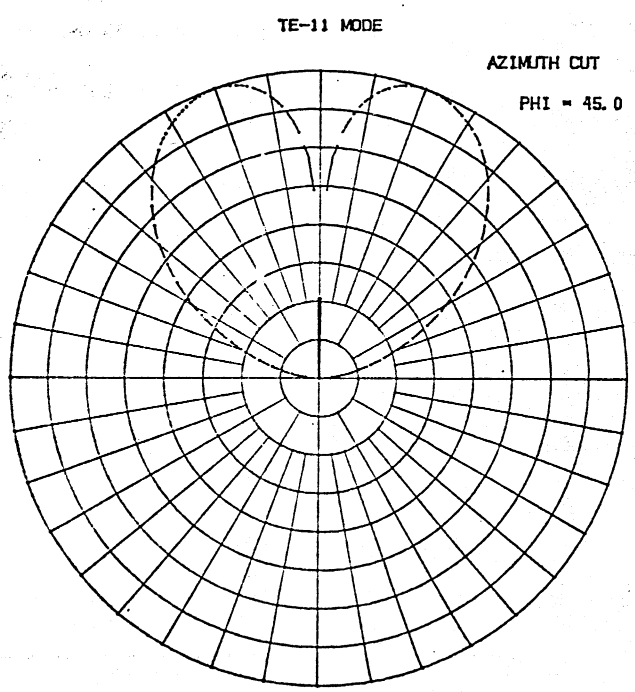
0 dB Response Everywhere

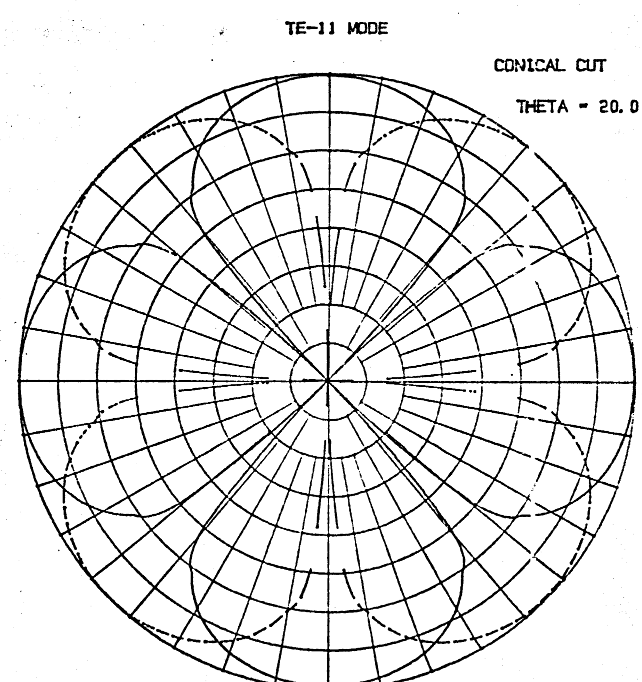


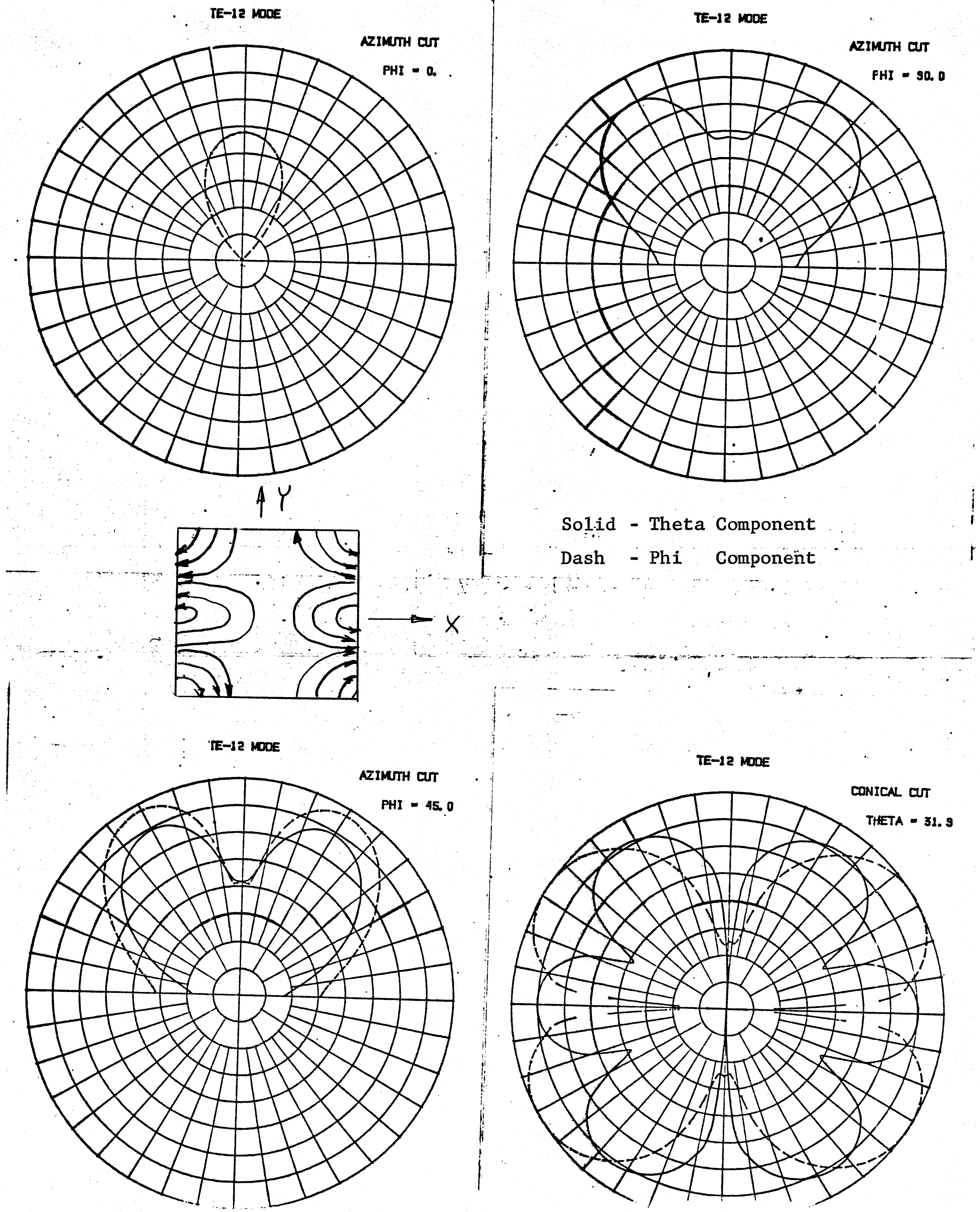


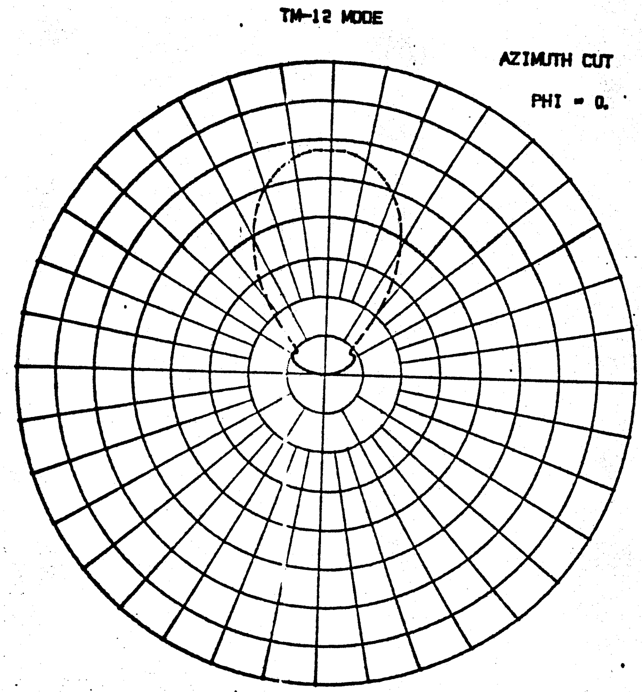


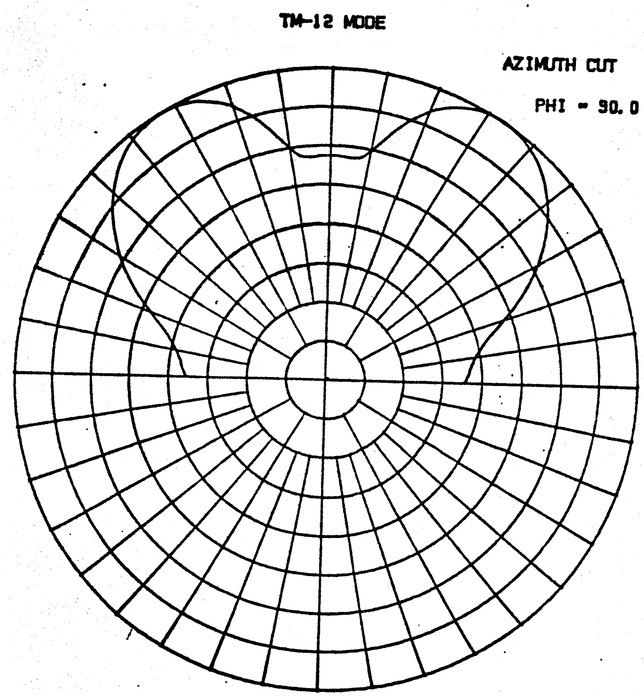
Solid - Theta Component
Dash - Phi Component

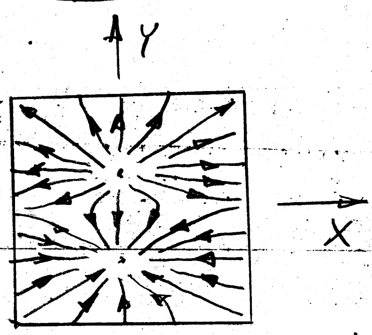




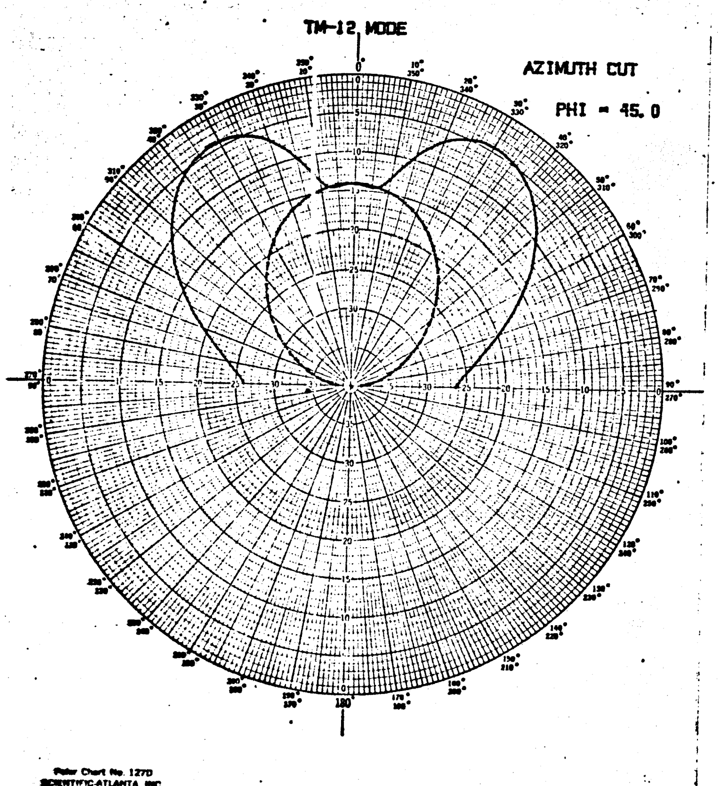

TE-12 MODE
AZIMUTH CUT
PHI = 0.
TE-12 MODE
AZIMUTH CUT
PHI = 90.0
Y
X
Solid - Theta Component
Dash - Phi Component
TE-12 MODE
AZIMUTH CUT
PHI = 45.0
TE-12 MODE
CONICAL CUT
THETA = 31.9

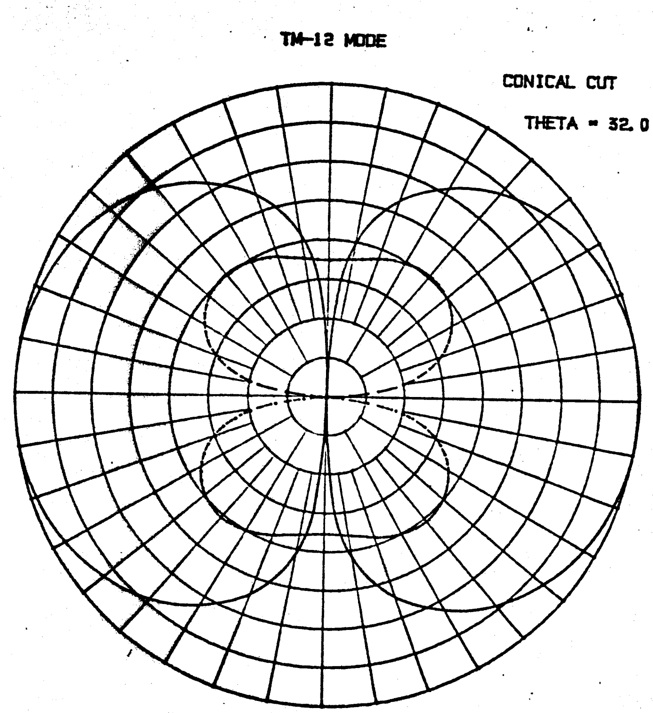




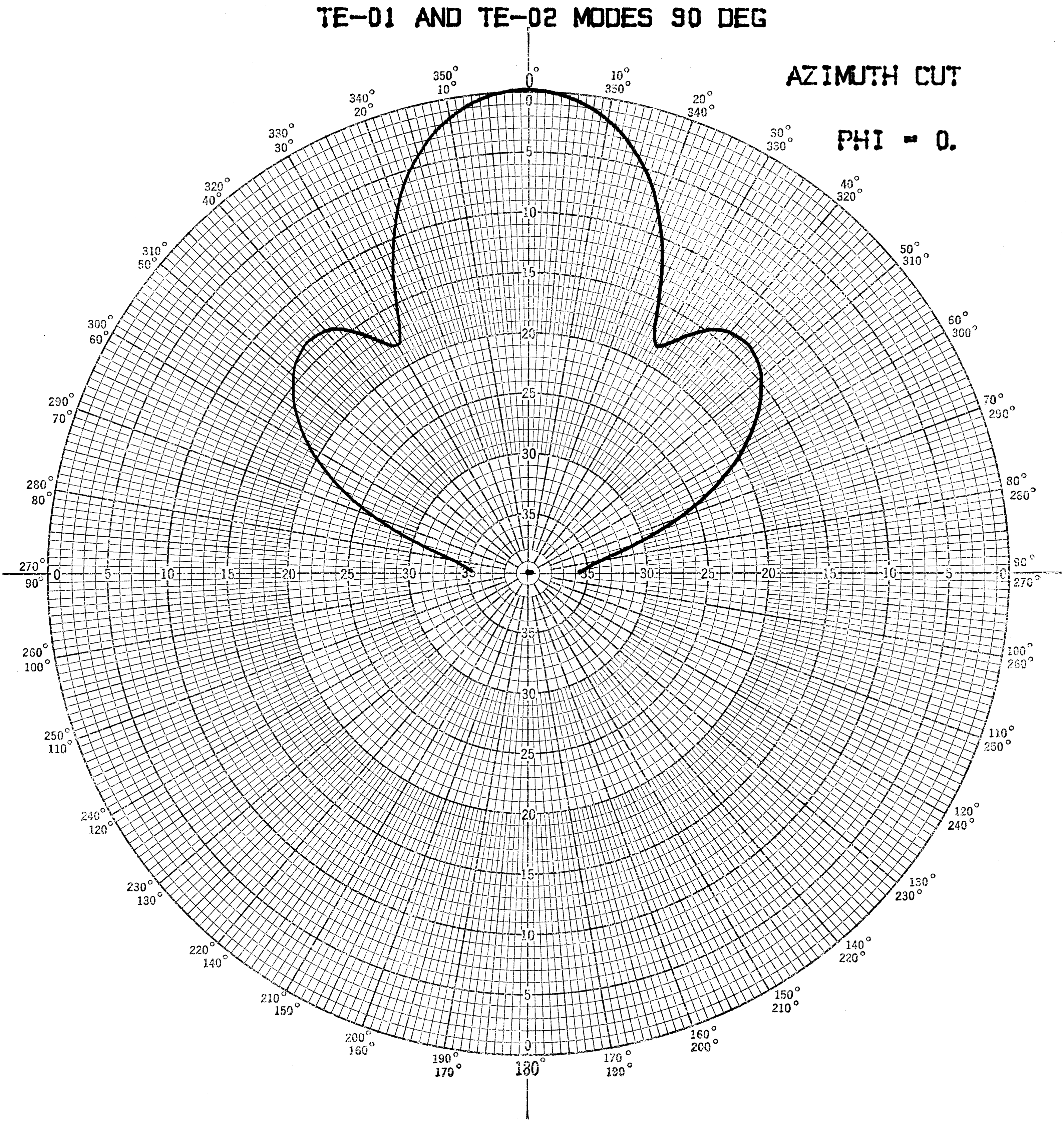


Solid - Theta Component
Dash - Phi Component





E Plane



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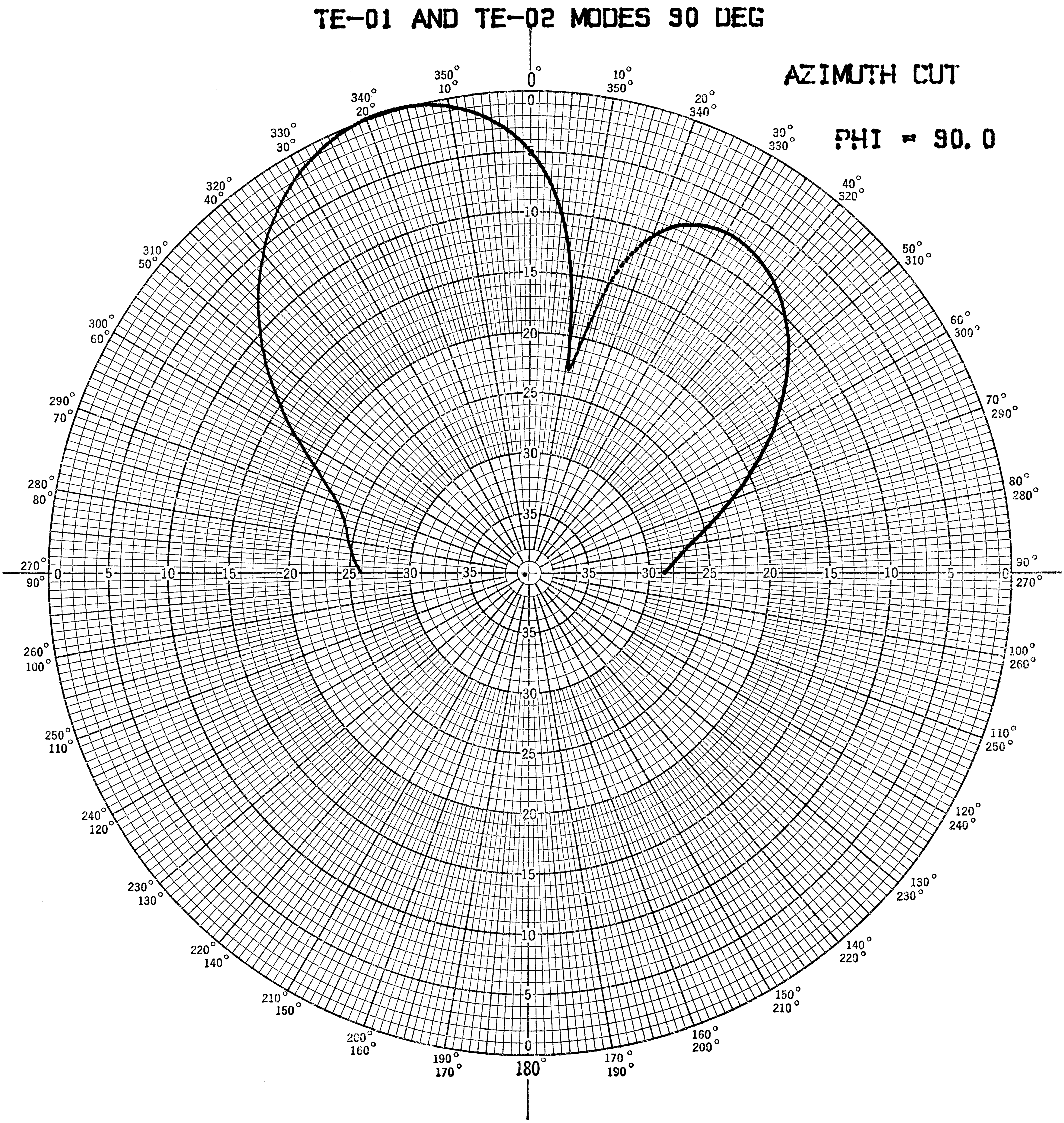
H Plane


TE-01 AND TE-02 MODES 90 DEG
AZIMUTH CUT
PHI = 90.0
Polar Chart No. 127D
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side of the pattern. The E plane pattern is unchanged from the E plane pattern of the TE_{01} mode horn because there is no E plane pattern response of the TE_{02} mode horn to horizontal or vertical polarization. If we did not know before hand that the TE_{02} mode was in the horn, we could determine this from the fact the E plane pattern was unchanged from the TE_{01} mode pattern. If the phase between the two modes was 270 degrees instead of 90 degrees as was drawn, then the null in the H plane pattern would occur on the left side of the pattern.

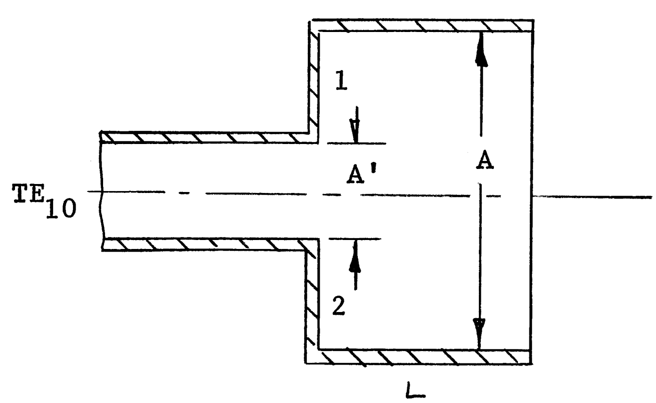
The TM_{11} is an interesting pattern because the response of the horn only has a theta component. When mounted on a model tower positioner, it will always be horizontally polarized.

BOX HORN

The box horn is an antenna which uses multimoding of the waveguide to an advantage. The aperture efficiency of a rectangular horn is reduced because the H plane has a cosine aperture distribution.

The maximum aperture efficiency is obtained with a uniform amplitude distribution in the aperture. Suppose there is an aperture collecting energy from a passing electromagnetic wave. The maximum energy which can be collected from a small portion of the wave is the total energy passing in that area. This corresponds to the peak amplitude response of the aperture less the losses in the antenna due to reflection mismatches and I^2R losses. If the amplitude response of the aperture some where else is reduced from that of the maximum response of the aperture, then that portion will collect less energy from the passing uniform plane wave. The amplitude response in that portion can be reduced by adding loss to that portion or reflecting the energy and re-radiating it. The antenna with the best aperture efficiency reflects the least amount of energy when illuminated with a plane wave. Hence a uniform aperture distribution will have the best aperture efficiency when equally matched at the input as an aperture with a nonuniform aperture distribution.

The box horn purposely introduces a discontinuity in the throat of the horn which generates the TE_{30} mode to satisfy the boundary conditions. The size of the horn is increased at the discontinuity to support the mode in propagation. Below is the H plane cut of the horn.



The E plane aperture is the same size as the waveguide or is flared from the input waveguide to prevent higher order modes in this plane. The width of the aperture, A, is designed so that the TE_{10} and TE_{30} modes can propagate but not the TE_{50} mode. Because there is symmetry about the center line of the discontinuity, only odd order modes are generated. The even order modes such as TE_{20} have odd mode symmetry across the center line and are not excited. No propagating TM modes are generated because the limited height of the horn cannot support the TM_{11} mode, the lowest order mode. At the aperture the distribution is a sum of the TE_{10} and TE_{30} modes.

$$E_y(x) = a_1 \cos\left(\frac{\pi x}{A}\right) e^{-j\beta_{10} L} + a_3 \cos\left(\frac{3\pi x}{A}\right) e^{-j\beta_{30} L}$$

Both modes propagate from the discontinuity where they started in phase in order to cancel the tangential electric fields on walls 1 and 2 in the figure above. To totally cancel the field on these walls many higher order modes had to be generated but only the TE_{10} and TE_{30} modes are able to propagate to the aperture. In the expression above β_{10} is the waveguide propagation constant of the TE_{10} mode and β_{30} is the propagation constant of the TE_{30} mode.

The amplitude distribution in the H plane will be more nearly uniform if the phase between the two modes is 180 degrees. Since the two modes will have different phase velocities depending on their respective cut off frequencies, the length, L, may be designed to give a difference in phase of 180 degrees. The length is given from these equations.

$$(\beta_{10} - \beta_{30}) L = \pi$$

Where
$$\beta_{10} = \frac{2\pi}{\lambda}\left[1 - \left(\frac{\lambda}{2A}\right)^2\right]^{1/2}$$

$$\beta_{30} = \frac{2\pi}{\lambda}\left[1 - \left(\frac{3\lambda}{2A}\right)\right]^{1/2}$$

From these relations we obtain an equation for the length L.

$$L = \frac{\lambda/2}{\left[1 - \left(\frac{\lambda}{2A}\right)^2\right]^{1/2} - \left[1 - \left(\frac{3\lambda}{2A}\right)^2\right]^{1/2}}$$

The difference in amplitudes between the two modes depends on the ratio of the input waveguide width to the aperture width. Silver, Microwave Antenna Theory and Design , gives the following integral formula for the ratio of the amplitudes in the two modes.

$$\frac{a_3}{a_1} = \frac{\int_{-A/2}^{A/2} \cos\left(\frac{\pi x}{A'}\right) \cos\left(\frac{3\pi x}{A}\right) dx}{\int_{-A/2}^{A/2} \cos\left(\frac{\pi x}{A'}\right) \cos\left(\frac{\pi x}{A}\right) dx}$$

This is plotted on page 254 versus the ratio of A'/A. It is found that the maximum gain occurs for $a_3/a_1 = 0.35$. The H plane beamwidth is limited to greater than 36 degrees because for smaller beamwidths the aperture A must be increased and the guide can then support the TE_{50} mode which is generated in the step but remains non-propagating.

For an example let us design a box horn at 2.2 GHz with a 45 degree 10 dB beamwidth. From a graph in Silver page 379, we find that the aperture width is 2.13 wavelengths for a ratio of $a_3/a_1 = 0.35$. The graph on page 254 gives the ratio of the input waveguide to the output aperture size to give an amplitude ratio of 0.35 . The ratio of A'/A is about 0.67. The input waveguide should be then 1.43 wavelengths wide. A guide this wide might be a problem because it will also support the TE_{20} mode as well as the desired TE_{10} mode. One solution would be to flare the waveguide after the original mode is launched in the waveguide. This would give us some phase error in the aperture but it would be minor.

Because we have been only dealing in wavelengths, we can continue to do so when finding the length of the box between the waveguide and the aperture.

$$L = \frac{\lambda/2}{\left[1 - \left(\frac{1}{2(2.13)}\right)^2\right]^{1/2} - \left[1 - \left(\frac{3}{2(2.13)}\right)^2\right]^{\frac{1}{2}}}$$

$$L = 1.91\,\lambda$$

Now we can substitute the frequency, find the wavelength, and the final dimensions of the design.

$$\lambda = \frac{11.80285 \times 10^9}{2.2 \times 10^9} = 5.365$$

$$L = 10.24 \text{ IN.}$$

$$A = 2.13\,\lambda = 11.43$$

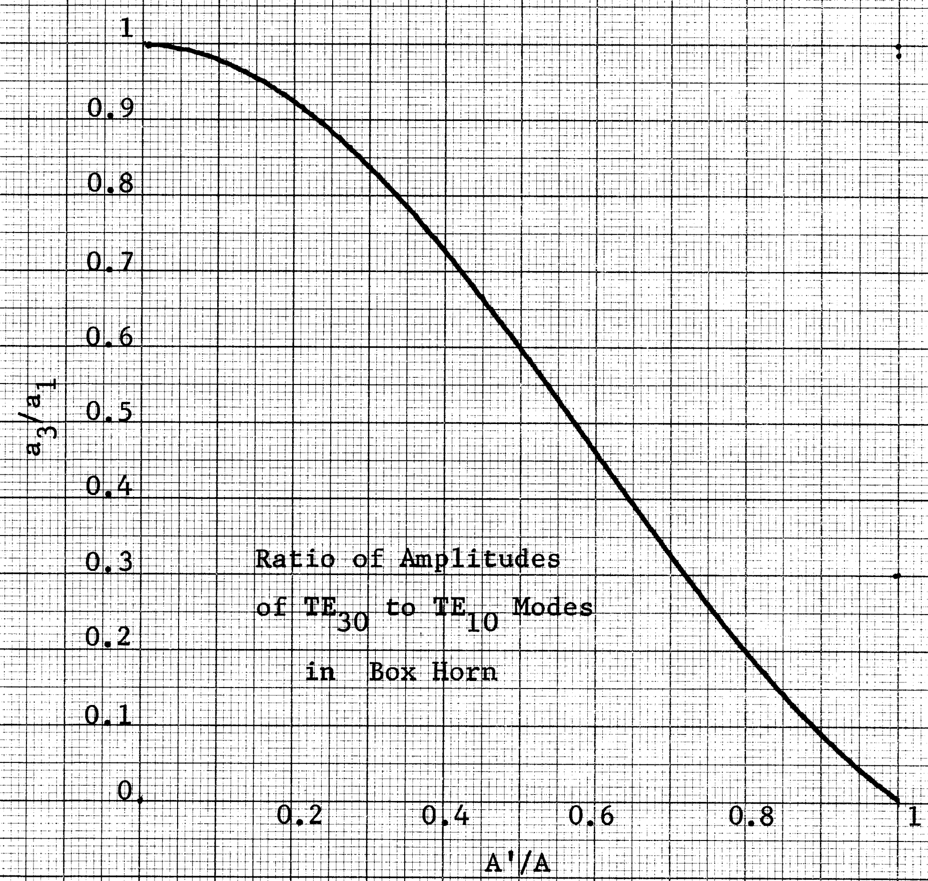
$$A' = 1.43\,\lambda = 7.67$$

The H plane pattern of this design is plotted on page 255.

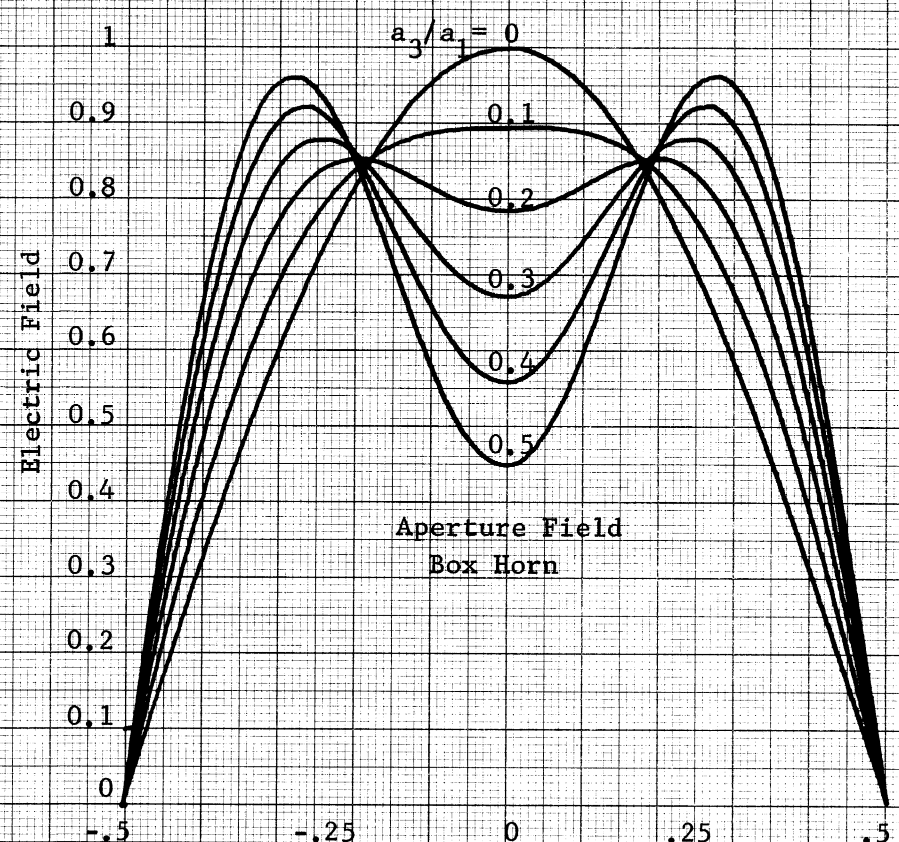
```
--
□
*ON,FMGR
:RU,OPL
  NO       A'/A        A3/A1

    2        .05   $$$$$$$$$
       OPL     47075
  ..END  OPL    ABORTED
OPL    ABORTED
:EX
 $END FMGR
L
*LU,1,9
**
□
*ON,FMGR
:RU,OPL
  NO       A'/A        A3/A1

    2        .05     .995333
    3        .10     .981420
    4        .15     .958518
    5        .20     .927051
    6        .25     .887601
    7        .30     .840890
    8        .35     .787772
    9        .40     .729208
   10        .45     .666245
   11        .50     .600000
   12        .55     .531626
   13        .60     .462295
   14        .65     .393170
   15        .70     .325378
   16        .75     .259992
   17        .80     .198002
   18        .85     .140301
   19        .90     .087663
   20        .95     .040730
 X-Y PLOTTER,  NEW PAPER? N
 MIN X =     .0000E+00     MAX X =     .1000E+01
 MIN Y =     .0000E+00     MAX Y =     .1000E+01
 ENTER COORDINATES OF LOWER LEFT CORNER 0,0
 ENTER COORDINATES OF UPPER RIGHT CORNER 1,1
 ENTER NUMBER OF INTERVALS BETWEEN POINTS 4
 ENTER PLOT PATTERN 1
 TO START PLOT ENTER INTERPOLATION ORDER 3
 DO YOU WANT GRID LINES? N
 PREPARE FOR SECOND GRAPH AND RESPOND U
:
```

BOX HORN

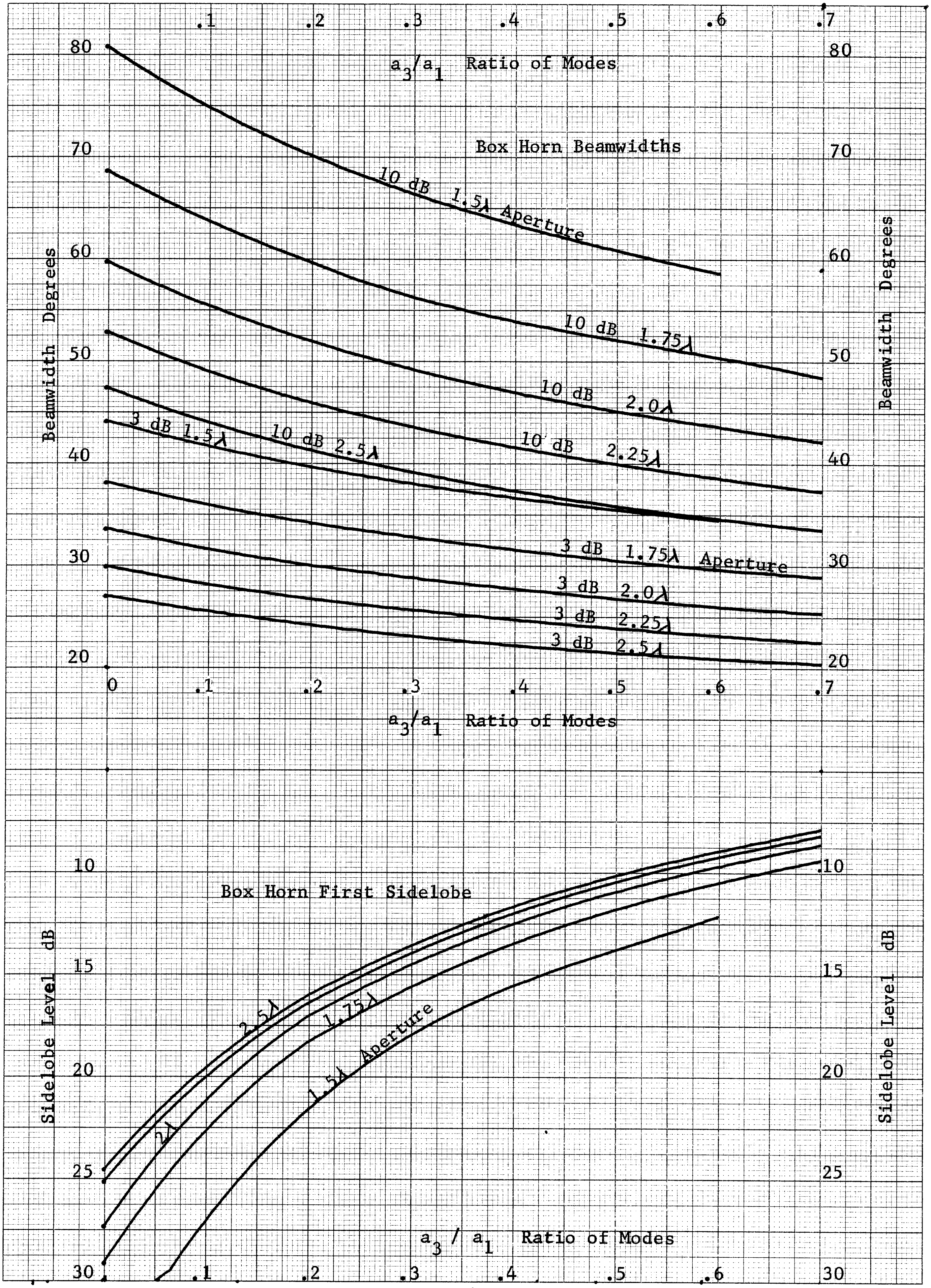
CURVE



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
a_3/a_1
Ratio of Amplitudes
of TE_{30} to TE_{10} Modes
in Box Horn
0.2
0.4
0.6
0.8
1
A'/A

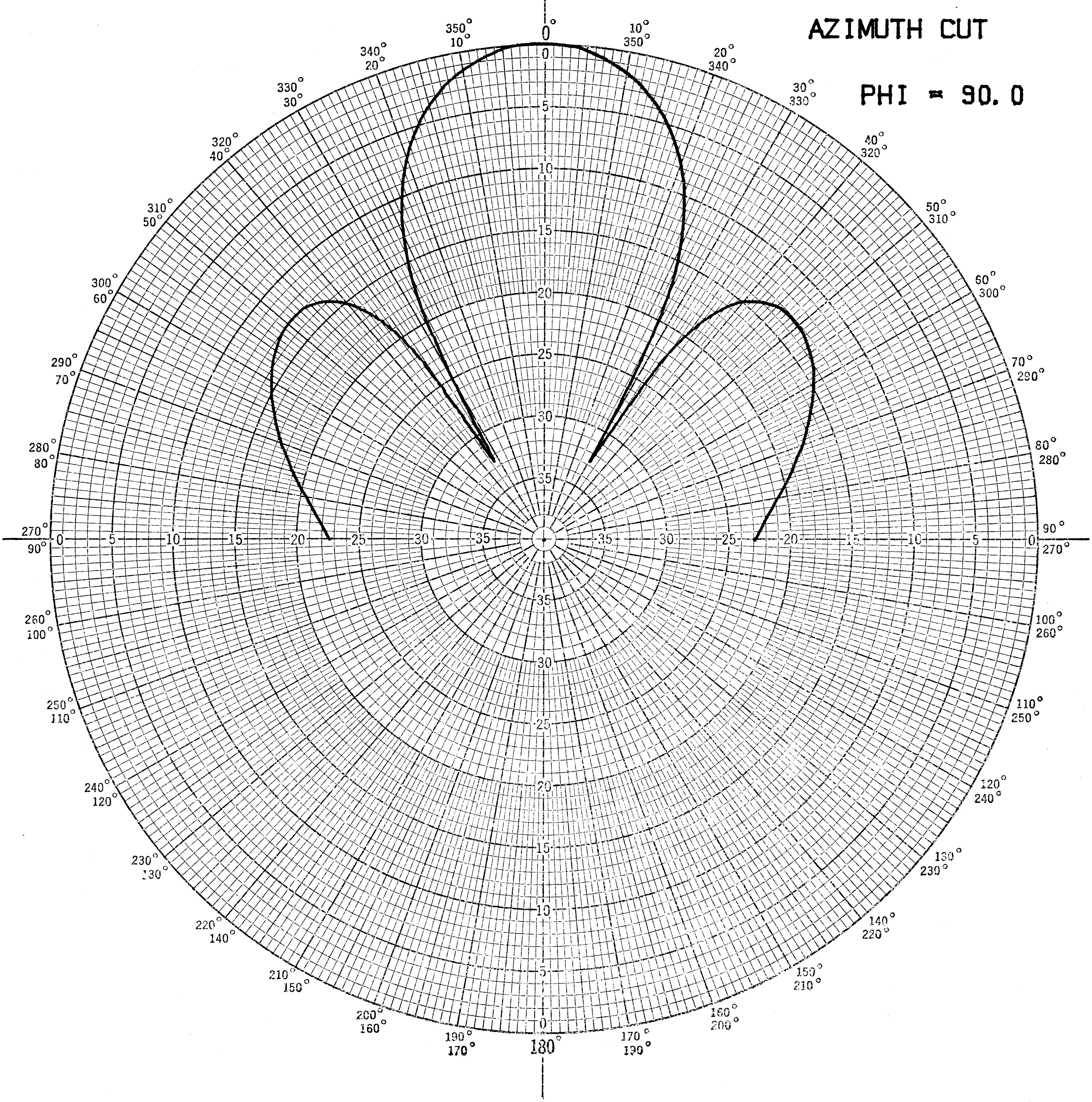


$a_3/a_1 = 0$
0.1
0.2
0.3
0.4
0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Electric Field
Aperture Field
Box Horn
-.5
-.25
0
.25
.5



a3/a1 Ratio of Modes
Box Horn Beamwidths
Beamwidth Degrees
10 dB 1.5λ Aperture
10 dB 1.75λ
10 dB 2.0λ
10 dB 2.25λ
10 dB 2.5λ
3 dB 1.5λ
3 dB 1.75λ Aperture
3 dB 2.0λ
3 dB 2.25λ
3 dB 2.5λ
0 .1 .2 .3 .4 .5 .6 .7
80 70 60 50 40 30 20
Box Horn First Sidelobe
Sidelobe Level dB
2.5λ
2λ
1.75λ
1.5λ Aperture
10 15 20 25 30
a3 / a1 Ratio of Modes

BOX HORN 11.43 APERTURE AT 2.2 GHZ A3/A1 – 0.35



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CIRCULAR COORDINATES

When the boundaries are coincident with the cylindrical coordinates, it is convenient to solve for the TE and TM modes in cylindrical coordinates. The scalar Helmholtz equation in circular coordinates is found from the equation on page 28.

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + \beta^2 \psi = 0$$

Again we will use the method of separation of variables to solve this equation.

$$\psi(\rho, \phi, z) = R(\rho)\, \Phi(\phi)\, Z(z)$$

When we substitute this in the equation and divide by ψ, we get

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\rho^2 \Phi}\frac{d^2 \Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + \beta^2 = 0$$

The third term can be separated out because it is independent of ρ and ϕ

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\beta_z^2$$

where β_z is the Z directed propagation constant. We need to substitute this back into the equation and multiply by ρ^2.

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2 \Phi}{d\phi^2} + (\beta^2 - \beta_z^2)\rho^2 = 0$$

Now the second term is independent of ρ and the only term with a ϕ dependence; it may be equated to a constant.

$$\frac{1}{\Phi}\frac{d^2 \Phi}{d\phi^2} = -n^2$$

We will define the third separation constant as

$$\beta_\rho^2 = \beta^2 - \beta_z^2$$

Then the differential equation for the radial component becomes

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \left((\beta_\rho \rho)^2 - n^2\right) R = 0$$

The solution to the Z(z) component will be a harmonic function which since we pick z as the direction of propagation becomes.

$$e^{-j\beta_z z}, \quad e^{j\beta_z z}$$

The differential equation of the ϕ component also has harmonic function solutions which we will pick as standing waves.

$$\sin n\phi, \quad \cos n\phi$$

If the whole circle is free of boundaries, then the equations of ϕ must be continuous at 0 and 2π. This condition is satisfied only if the separation constant n is an integer.

The R component differential equation is called the Bessel's equation and has solutions called Bessel functions. These equations are denoted in general as $Z_n(\beta_\rho \rho)$. The number of the Bessel function is dependent on the separation constant of ϕ.

Bessel functions

$J_n(x)$	Standing Wave		
$N_n(x)$ or $Y_n(x)$	Standing Wave	(infinite at x = 0)	Neumann Function
$H_n^{(1)}(x)$	Traveling Wave	similiar to $e^{j\beta_\rho \rho}$	Hankel Function
$H_n^{(2)}(x)$	Traveling Wave	similiar to $e^{-j\beta_\rho \rho}$	Hankel Function

The Bessel function required will depend on the boundary conditions. The first few Bessel functions $J_n(x)$ are plotted on page 258. These are the functions we will be using the most. They are tabulated functions like sine and cosine.

CIRCULAR TE AND TM MODES

We will derive TE and TM modes in cylindrical coordinates by using Z directed vector potentials.

For TE mode: $\bar{F} = \bar{a}_z \psi$

$$\bar{E} = -\nabla \times \bar{F} \qquad \bar{H} = -j\omega\epsilon \bar{F} + \frac{\nabla(\nabla \cdot \bar{F})}{j\omega\mu}$$

From the equations on pages 27 and 28 the field equations are given in cylindrical coordinates for TE modes by:

$$E_\rho = -\frac{1}{\rho}\frac{\partial \psi}{\partial \phi} \qquad H_\rho = \frac{1}{j\omega\mu}\frac{\partial^2 \psi}{\partial \rho\, \partial z}$$

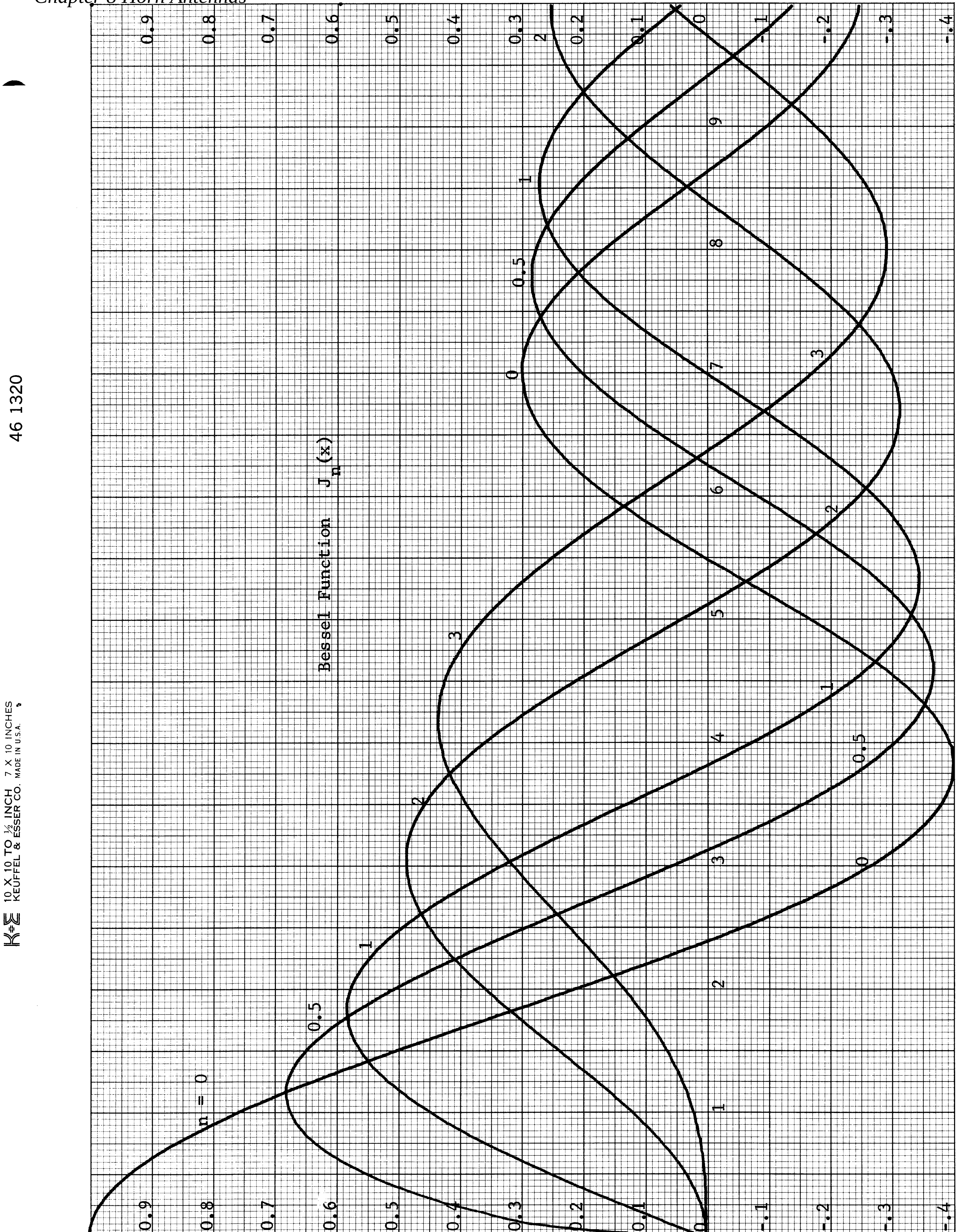
$$E_\phi = \frac{\partial \psi}{\partial \rho} \qquad H_\phi = \frac{1}{j\omega\mu\rho}\frac{\partial^2 \psi}{\partial \phi\, \partial z}$$

TE Mode
(H Mode)

$$E_z = 0 \qquad H_z = \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)\psi$$

Similiarly the TM modes are derived from a Z directed magnetic vector potential.

For TM modes: $\bar{A} = \bar{a}_z \psi$



Bessel Function $J_n(x)$
n = 0
0.5
1
2
3

The fields are found for the TM modes by the equations:

$$\bar{H} = \nabla \times \bar{A} \qquad \bar{E} = -j\omega\mu \bar{A} + \frac{\nabla(\nabla \cdot \bar{A})}{j\omega\epsilon}$$

When expanded these become:

$$E_\rho = \frac{1}{j\omega\epsilon}\frac{\partial^2 \psi}{\partial\rho\,\partial z} \qquad H_\rho = \frac{1}{\rho}\frac{\partial \psi}{\partial \phi}$$

$$E_\phi = \frac{1}{j\omega\epsilon\rho}\frac{\partial^2 \psi}{\partial\phi\,\partial z} \qquad H_\phi = -\frac{\partial \psi}{\partial \rho}$$

TM Modes

(E Modes)

$$E_z = \frac{1}{j\omega\epsilon}\left(\frac{\partial}{\partial z^2} + \beta^2\right)\psi \qquad H_z = 0$$

CIRCULAR WAVEGUIDES

We will use the above equations to find the possible modes in the circular waveguide. The direction of propagation is in the Z axis direction so we have traveling waves in that direction with β_z as the waveguide propagation constant. Since there is no bound on ϕ, the separation constant n will be an integer. There is no loss in generality if only the cosine term is used. The Neumann function $N_n(\beta_\rho \rho)$ is infinite at $\rho = 0$ so we must eliminate it as a possible solution. The vector potential function is reduced to

$$\psi = J_n(\beta_\rho \rho)\cos n\phi\, e^{-j\beta_z z}$$

TE MODES

For TE modes the tangential electric field must be zero at ρ = a (the radius of the waveguide). The tangential electric field is

$$E_\phi = \frac{\partial \psi}{\partial \rho} = \beta_\rho J_n'(\beta_\rho \rho)\cos(n\phi)\, e^{-j\beta_z z}$$

The electric field will be zero when $J_n'(\beta_\rho a) = 0$. Given n there will be an infinite number of zeros of the derivative of the Bessel function which will determine the mode numbers. The first few zeros of the Bessel function derivative, $J_n'(x)$, are given in the table below.

Zeros of $J_n'(x)$

Bessel Function order

p \ n	0	1	2	3
1	3.832	1.841	3.054	4.201
2	7.016	5.331	6.706	8.015
3	10.173	8.536	9.969	11.346

X'_{np}

This determines the radial separation constant.

$$\beta_\rho a = \chi'_{np}$$

The propagation constant in the Z direction is found from the equation:

$$\beta_z^2 = \beta^2 - \beta_\rho^2 = \beta^2 - \left(\frac{\chi'_{np}}{a}\right)^2$$

The cut off frequency is determined from the point where $\beta_z = 0$

$$\frac{\chi'_{np}}{a} = \frac{2\pi}{\lambda_c} \qquad \lambda_c = \frac{2\pi a}{\chi'_{np}}$$

From the table on page 259 we determine that the maximum cut off wavelength occurs for n = 1 and p = 1

$$\lambda_c = \frac{2\pi a}{1.841} = 3.413\,a$$

TE_{11} Mode

$$F_c = \frac{3.458}{a}\ GHz\text{-}IN$$

TM MODES

Again for TM modes we must satisfy zero tangential electric field at the boundary ρ = a. The TM mode has a E_z component given by

$$E_z = \frac{1}{j\omega\epsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)\psi$$

This will be zero on the walls if $\psi = 0$ at $\rho = a$ which implies

$$J_n(\beta_\rho a) = 0$$

The radial separation constant, β_ρ, will be determined by the zeros of the Bessel function. The following table is the first few zeros of the Bessel function $J_n(x)$.

Bessel Function Order

	p \ n	0	1	2	3
Zero	1	2.405	3.832	5.136	6.380
Number	2	5.520	7.016	8.417	9.761
	3	8.654	10.173	11.620	13.015

X_{np}

$$\beta_\rho a = \chi_{np}$$

The propagation constant in the Z direction is found from

$$\beta_z^2 = \beta^2 - \beta_\rho^2 = \beta^2 - \left(\frac{\chi_{np}}{a}\right)^2$$

The cut off frequency is that point where $\beta_z = 0$.

$$\lambda_{c_{TM}} = \frac{2\pi a}{\chi_{np}}$$

The lowest order TM mode in circular waveguide is the TM_{01} mode.

$$\lambda_{c_{TM_{01}}} = 2.612\ a$$

$$F_{c_{TM_{01}}} = 4.519/\ a \quad \text{GHz-in.}$$

The order of the circular waveguide modes is TE_{11}, TM_{01}, TE_{21}, (TE_{01}, TM_{11}). The TE_{01} and TM_{11} modes have the same cut off frequency. The TE_{11}, TM_{01}, and TE_{01} modes are useful for feeding waveguide horns.

The fields of the TE mode in circular waveguide are found from the Z directed electric vector potential.

$$\psi_{TE} = J_n(\beta_\rho \rho)\cos(n\phi)e^{-j\beta_z z} \qquad \beta_\rho = \frac{\chi'_{np}}{a}$$

Using the equations on page 257, we find the waveguide fields.

TE Mode

$$E_\rho = \frac{n}{\rho}\ J_n(\beta_\rho \rho)\sin(n\phi)\ e^{-j\beta_z z}$$

$$E_\phi = \beta_\rho J_n'(\beta_\rho \rho)\cos(n\phi)\ e^{-j\beta_z z}$$

$$H_\rho = -\frac{\beta_z \beta_\rho}{\omega\mu}\ J_n'(\beta_\rho \rho)\cos(n\phi)\ e^{-j\beta_z z}$$

$$H_\phi = \frac{\beta_z n}{\omega\mu\rho}\ J_n(\beta_\rho \rho)\sin(n\phi)\ e^{-j\beta_z z}$$

$$H_z = \frac{1}{j\omega\mu}(\beta^2 - \beta_z^2)J_n(\beta_\rho\rho)\cos(n\phi)\ e^{-j\beta_z z}$$

Similiarly, we can find the TM mode waveguide fields by using the Z directed magnetic vector potential function.

$$\psi_{TM} = J_n(\beta_\rho \rho)\cos(n\phi)\ e^{-j\beta_z z} \qquad \beta_\rho = \frac{\chi_{np}}{a}$$

This is the same potential function as the TE mode function. But the separation constant, β_ρ, is equal to the zero of the Bessel function, $J_n(x)$, divided by the radius, a, of the waveguide. We can find the waveguide fields by using the equations on page 259 for TM modes in circular coordinates.

TM modes of circular waveguide

TM Mode

$$E_\rho = \frac{-\beta_\rho \beta_z}{\omega\epsilon} J_n'(\beta_\rho \rho) \cos(n\phi)\, e^{-j\beta_z z}$$

$$E_\phi = \frac{n\beta_z}{\omega\epsilon\rho} J_n(\beta_\rho \rho) \sin(n\phi)\, e^{-j\beta_z z}$$

$$E_z = \frac{1}{j\omega\epsilon}(\beta^2 - \beta_z^2) J_n(\beta_\rho \rho) \cos(n\phi)\, e^{-j\beta_z z}$$

$$H_\rho = \frac{-n}{\rho} J_n(\beta_\rho \rho) \sin(n\phi)\, e^{-j\beta_z z}$$

$$H_\phi = -\beta_\rho J_n'(\beta_\rho \rho) \cos(n\phi)\, e^{-j\beta_z z}$$

$$H_z = 0$$

We should also find the Z directed wave impedance of the circular waveguide.

$$(Z_o)_{np} = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho}$$

When we substitute from the equations above for the TE and TM modes, we find

$$(Z_o)_{np}^{TE} = \frac{\omega\mu}{\beta_z}$$

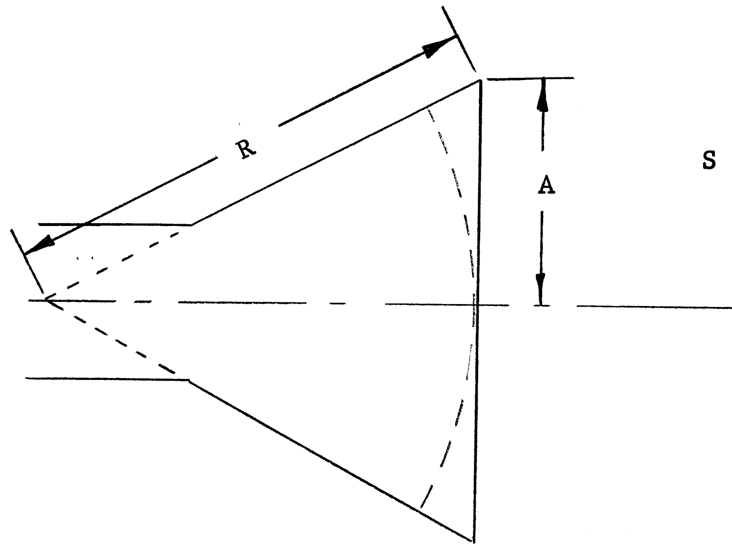
$$(Z_o)_{np}^{TM} = \frac{\beta_z}{\omega\epsilon}$$

These are the same equations that were found on page 242 for the rectangular waveguide. The TE mode impedance becomes infinite as the cut off frequency is approached and the TM wave impedance approaches zero. For very large diameter waveguides the wave impedances approach that of free space.

CIRCULAR WAVEGUIDE HORNS

We use the equations for the fields in circular waveguides to find the patterns of circular waveguide horns. Again as with rectangular horns we assume that the flare angle of the horn is gradual enough that the higher modes generated are insignificant at the aperture plane. The amplitude distribution at the horn aperture is the same as the mode exciting it. But there will be a quadratic phase taper in the aperture because the waves at the edges of the aperture must travel further than those traveling to the center.

Let us first consider the geometry of the circular waveguide horn regardless of the mode in the waveguide.



$$S = \frac{A^2}{2R\lambda}$$

The maximum phase error distance in the aperture is denoted S which is given in terms of wavelengths in the figure above. Like the rectangular waveguide horn, this is the dimensionless parameter of various curves.

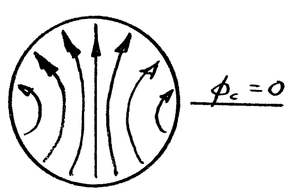
TE_{11} MODE

The TE_{11} mode is the dominant mode in circular waveguide. When we substitute into the general field equations on page 261, we get the following field equations.

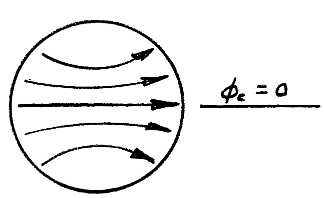
$$E_\rho = \frac{E_0}{\rho} J_1\left(\frac{\chi'_{11}\rho}{a}\right) \sin\phi \, e^{-j\beta_z z}$$

$$E_\phi = \frac{E_0 \chi'_{11}}{a} J_1'\left(\frac{\chi'_{11}\rho}{a}\right) \cos\phi \, e^{-j\beta_z z}$$

We will be using the Huygens source approximation in the aperture so these are the only components we will need. The electric field on the aperture is given below.



It will be convenient to rotate the horn so that $\phi_c = 0$ is aligned with the maximum electric field instead of $\phi_c = 90$ degrees.



When we have rotated the horn by 90 degrees, the aperture electric field becomes

$$E_\rho = \frac{E_0}{\rho} J_1\left(\frac{\chi_{11}'\rho}{a}\right) \cos\phi_c \qquad \chi_{11}' = 1.841$$

$$E_{\phi_c} = \frac{-E_0\chi_{11}'}{a} J_1'\left(\frac{\chi_{11}'\rho}{a}\right) \sin\phi_c$$

We have started using ϕ_c for the cylindrical coordinate Phi component to prevent confusion with ϕ of spherical coordinates. Most of the time they are identical.

TRANSFORMATION BETWEEN VECTORS IN CYLINDRICAL AND SPHERICAL COORDINATES

To perform this we will need the vector dot products between the various components.

$$\bar{a}_\rho \cdot \bar{a}_r = \sin\theta(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c) = \sin\theta\cos(\phi - \phi_c)$$
$$\bar{a}_\rho \cdot \bar{a}_\theta = \cos\theta(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c) = \cos\theta\cos(\phi - \phi_c)$$
$$\bar{a}_\rho \cdot \bar{a}_\phi = \cos\phi\sin\phi_c - \sin\phi\cos\phi_c = -\sin(\phi - \phi_c)$$

$$\bar{a}_{\phi_c} \cdot \bar{a}_r = \sin\theta(\sin\phi\cos\phi_c - \cos\phi\sin\phi_c) = \sin\theta\sin(\phi - \phi_c)$$
$$\bar{a}_{\phi_c} \cdot \bar{a}_\theta = \cos\theta(\sin\phi\cos\phi_c - \cos\phi\sin\phi_c) = \cos\theta\sin(\phi - \phi_c)$$
$$\bar{a}_{\phi_c} \cdot \bar{a}_\phi = \cos\phi\cos\phi_c + \sin\phi\sin\phi_c = \cos(\phi - \phi_c)$$

$$\bar{a}_z \cdot \bar{a}_r = \cos\theta \qquad \bar{a}_z \cdot \bar{a}_\theta = -\sin\theta \qquad \bar{a}_z \cdot \bar{a}_\phi = 0$$

If $\phi = \phi_c$, then the products reduce to

$$\bar{a}_\rho \cdot \bar{a}_r = \sin\theta \qquad \bar{a}_\rho \cdot \bar{a}_\theta = \cos\theta \qquad \bar{a}_\rho \cdot \bar{a}_\phi = 0$$
$$\bar{a}_{\phi_c} \cdot \bar{a}_r = 0 \qquad \bar{a}_{\phi_c} \cdot \bar{a}_\theta = 0 \qquad \bar{a}_{\phi_c} \cdot \bar{a}_\phi = 1$$
$$\bar{a}_z \cdot \bar{a}_r = \cos\theta \qquad \bar{a}_z \cdot \bar{a}_\theta = -\sin\theta \qquad \bar{a}_z \cdot \bar{a}_\phi = 0$$

When integrating on an aperture, the vectors are not referenced to the same point and $\phi_c \neq \phi$. The vector magnitudes are found using these dot products.

$$U_r = \bar{a}_r \cdot (U_\rho \bar{a}_\rho + U_{\phi_c}\bar{a}_{\phi_c} + U_z \bar{a}_z)$$

$$U_r = U_\rho \cos\theta(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c)$$
$$\quad + U_{\phi_c}\sin\theta(\sin\phi\cos\phi_c - \cos\phi\sin\phi_c) + U_z\cos\theta$$

$$U_\theta = U_\rho \cos\theta(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c)$$
$$\quad + U_{\phi_c}\cos\theta(\sin\phi\cos\phi_c - \cos\phi\sin\phi_c) - U_z\sin\theta$$

$$U_\phi = U_\rho(\cos\phi\sin\phi_c - \sin\phi\cos\phi_c) + U_{\phi_c}(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c)$$

For completeness we must include the transformation of the components when going from spherical coordinates to cylindrical coordinates.

$$U = U_r \sin\theta(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c) + U_\theta\cos\theta(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c) + U_\phi(\cos\phi\sin\phi_c - \sin\phi\cos\phi_c)$$

$$U_{\phi_c} = U_r\sin\theta(\sin\phi\cos\phi_c - \cos\phi\sin\phi_c) + U_\theta\cos\theta(\sin\phi\cos\phi_c - \cos\phi\sin\phi_c) + U_\phi(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c)$$

$$U_z = U_r\cos\theta - U_\theta\sin\theta$$

Using the Huygens source method, we can find the radiation intensity from the circular horn. There will be a constant factor in all the expressions for the far field.

$$\text{Let } \xi = \frac{j e^{-j\beta r}(1+\cos\theta)}{2\lambda r}$$

Then we can find the theta and phi components for the fields by using the transformation relations between cylindrical and spherical coordinates.

$$E_\theta = \xi\int_0^{2\pi}\int_0^{a}\Big(\frac{E_0}{\rho}J_1\Big(\frac{\chi'_{11}\rho}{a}\Big)\cos\phi_c(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c) - \frac{E_0\chi'_{11}}{a}J_1'\Big(\frac{\chi'_{11}\rho}{a}\Big)\sin\phi_c(\sin\phi\cos\phi_c - \sin\phi_c\cos\phi)\Big) * \rho\, e^{j\beta\rho\sin\theta\cos(\phi-\phi_c)}\, e^{-j\frac{\pi\rho^2}{\lambda R}}\, d\rho\, d\phi_c$$

The distance from a point on the aperture to the zero phase reference plane defined by the direction (θ, ϕ) is given as

$$\rho\sin\theta\cos(\phi - \phi_c)$$

The zero phase reference plane is through the center of the aperture.

The other far field component is given by this integral.

$$E_\phi = \xi\int_0^{2\pi}\int_0^{a}\Big(\frac{E_0}{\rho}J_1\Big(\frac{\chi'_{11}\rho}{a}\Big)\cos\phi_c(\cos\phi\sin\phi_c - \sin\phi\cos\phi_c) - \frac{E_0\chi'_{11}}{a}J_1'\Big(\frac{\chi'_{11}\rho}{a}\Big)\sin\phi_c(\cos\phi\cos\phi_c + \sin\phi\sin\phi_c) * \rho\, e^{j\beta\rho\sin\theta\cos(\phi-\phi_c)}\, e^{-j\frac{\pi\rho^2}{\lambda R}}\, d\rho\, d\phi_c$$

Now we need to consider the principle plane patterns.

E Plane $\phi = 0$

When we substitute this into the integrals, they become a little more manageable.

$$E_\theta = \xi E_0 \int_0^{2\pi}\int_0^a \left(J_1\left(\frac{\chi'_{11}\rho}{a}\right)\cos^2\phi_c + \frac{\chi'_{11}\rho}{a} J_1'\left(\frac{\chi'_{11}\rho}{a}\right)\sin^2\phi_c \right)$$
$$* \, e^{j\beta\rho\sin\theta\cos\phi_c} \, e^{-j\frac{\pi}{\lambda R}\rho^2} \, d\rho \, d\phi_c$$

$$E_\phi = \xi E_0 \int_0^{2\pi}\int_0^a \left(J_1\left(\frac{\chi'_{11}\rho}{a}\right) - \frac{\chi'_{11}\rho}{a} J_1'\left(\frac{\chi'_{11}\rho}{a}\right)\right)\sin\phi_c\cos\phi_c$$
$$* \, e^{j\beta\rho\sin\theta\cos\phi_c} \, e^{-j\frac{\pi\rho^2}{\lambda R}} \, d\rho \, d\phi_c$$

When the following integral is evaluated, we find that it is zero.

$$\int_0^{2\pi} \sin\phi\cos\phi \, e^{ju\cos\phi} \, d\phi = 0$$

$$\int_0^{2\pi} \sin\phi\cos\phi \, e^{ju\sin\phi} \, d\phi = 0$$

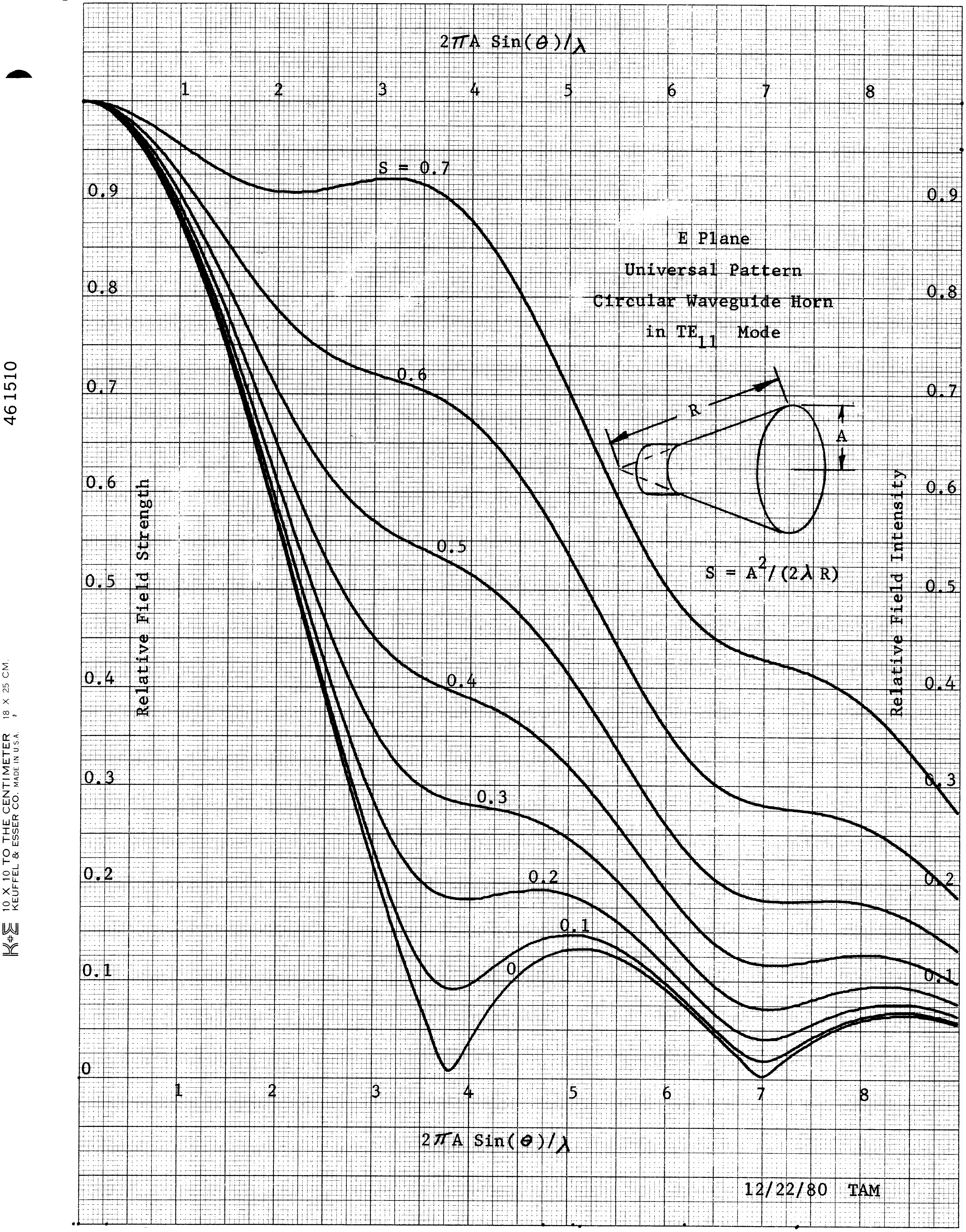
This is the form of the integral for the ϕ component of the far field in the E plane. Therefore ϕ is the cross polarization component in the E plane and it is zero. We can also see this from the electric field pattern in the aperture which is drawn on page 263. There is a symmetry about the $\phi_c = 0$ axis where the vertically directed electric fields are matched across this line and equally spaced from the zero reference plane.

H Plane $\phi = \pi/2$

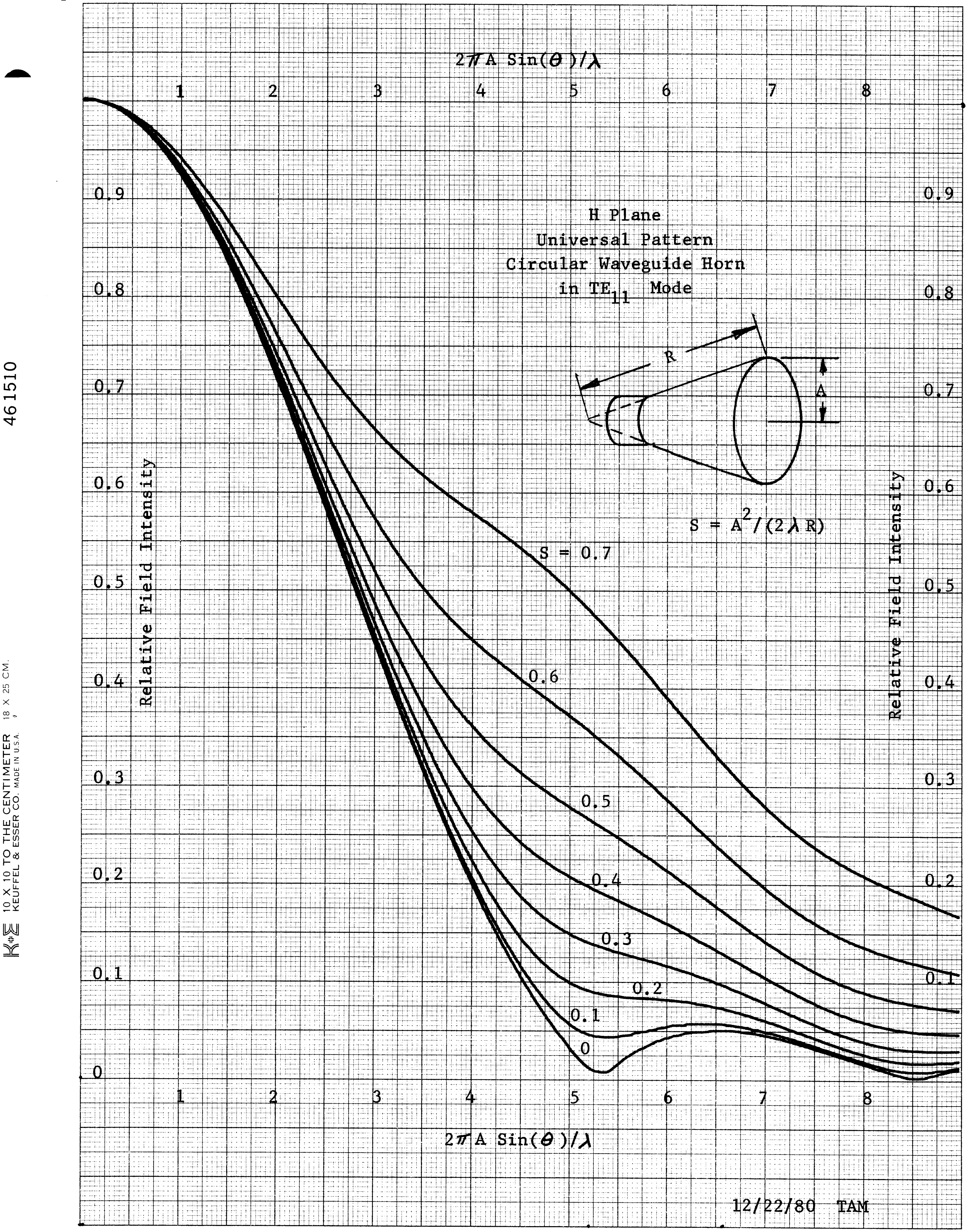
Again when we substitute this value of ϕ into the integrals they are simplified. The integral for the θ component has the same form as the integral of the ϕ component in the E plane. It will be zero also because of the integral over the ϕ_c variable. There is no cross polarization response.

$$E_\phi = -\xi E_0 \int_0^{2\pi}\int_0^a \left(J_1\left(\frac{\chi'_{11}\rho}{a}\right)\cos^2\phi_c + \frac{\chi'_{11}\rho}{a} J_1'\left(\frac{\chi'_{11}\rho}{a}\right)\sin^2\phi_c \right)$$
$$* \, e^{j\beta\rho\sin\theta\sin\phi_c} \, e^{-j\frac{\pi\rho^2}{\lambda R}} \, d\rho \, d\phi_c$$

By a suitable change of variables in the integrals, universal radiation patterns can be generated for the circular horn in the TE_{11} mode. These are plotted on pages 267 and 268 for the E and H planes, respectively. These curves are of the same form as the curves on pages 220 and 221 for the rectangular waveguide. Like those curves there is a dimensionless



2πA Sin(θ)/λ
1
2
3
4
5
6
7
8
S = 0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
E Plane
Universal Pattern
Circular Waveguide Horn
in TE11 Mode
R
A
S = A²/(2λR)
Relative Field Strength
Relative Field Intensity
12/22/80 TAM
461510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.



2π A Sin(θ)/λ
1
2
3
4
5
6
7
8
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Relative Field Intensity
H Plane
Universal Pattern
Circular Waveguide Horn
in TE11 Mode
R
A
S = A²/(2λR)
S = 0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
12/22/80 TAM
461510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
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parameter, S, which is the maximum aperture phase deviation in wavelengths. The two curves are no longer independent as were the rectangular horn curves, but have the same value for S. When values are taken off the curves to find the pattern, the factor $(1 + \cos\theta)/2$ must be multiplied by the value to find the pattern. Also remember that field intensity is proportional to voltage so the values off the curve must be squared to find the power pattern (20 Log( )).

GAIN (DIRECTIVITY)

As usual with horns we will assume that gain equals directivity since the losses are very small if the horn is well matched to the input connector. The peak of the gain occurs at $\theta = 0$. The first thing to calculate is the total power in the aperture.

$$\text{Power} = \frac{1}{\eta}\int_0^{2\pi}\int_0^{a}\left(|E_\rho|^2 + |E_{\phi_c}|^2\right)\rho\, d\rho\, d\phi_c$$

From the Huygens source approximation we have assumed that the ratio of the electric to magnetic field is the same as in free space in the aperture. When the fields are substituted into the integral, the integral over the ϕ_c can be performed easily and a change of variable can be made to reduce the integral to this form:

$$\text{POWER} = \frac{\pi E_0}{\eta}\int_0^1 \frac{J_1^2(\chi_{11}u)}{u} + J'^2_1(\chi'_{11}u)(\chi'_{11})^2 u\, du$$

This integral is most easily solved by numerical techniques. Now we must find the far field at $\theta = 0$.

$$E_\theta(0) = \xi E_0 \int_0^{2\pi}\int_0^{a}\left(J_1\left(\frac{\chi'_{11}\rho}{a}\right)\cos^2\phi_c + \frac{\chi'_{11}\rho}{a}J_1'\left(\frac{\chi'_{11}\rho}{a}\right)\sin^2\phi_c\right)e^{-j\frac{\pi\rho^2}{\lambda R}}\, d\rho\, d\phi_c$$

The integral over ϕ_c can be separated out of the integral.

$$\int_0^{2\pi}\cos^2\phi_c\, d\phi_c = \int_0^{2\pi}\sin^2\phi_c\, d\phi_c = \pi$$

We can make the following substitutions in the integral variables to reduce the integral to a dimensionless integral.

$$u = \frac{\rho}{a} \qquad du = \frac{d\rho}{a} \qquad S = \frac{a^2}{2\lambda R}$$

The integral for the field intensity at $\theta = 0$ is reduced to

$$E_\theta(0) = \pi\xi E_0 a\int_0^1\left(J_1(\chi'_{11}u) + \chi'_{11}u\, J_1'(\chi'_{11}u)\right)e^{-j2\pi S u^2}\, du$$

The maximum radiation intensity $= \dfrac{r^2\,|E_\theta(0)|^2}{\eta}$

The term $r^2\,|\xi(0)|^2 = 1./\lambda^2$

Using the maximum radiation intensity and the total power radiated, we can find the directivity from the formula

$$\text{Directivity} = \frac{4\pi U_{max}}{P_r}$$

When we substitute into this formula the maximum radiation intensity and the power radiated, we get a formula for the directivity of the circular horn in the TE_{11} mode.

$$\text{Directivity} = \left(\frac{2\pi a}{\lambda}\right)^2 \frac{\left|\int_0^1 \left(J_1(\chi'_{11}u) + \chi'_{11}u\, J'(\chi'_{11}u)\right)e^{-j2\pi S u^2}\,du\right|^2}{\int_0^1 \left(\frac{J_1^2(\chi'_{11}u)}{u} + J'^2(\chi'_{11}u)(\chi'_{11})^2 u\right)du}$$

The division of the two integrals only depends on S, the maximum aperture phase deviation. If we take the logarithm of the expression, we find that the gain (directivity) is proportional to the aperture size minus a gain correction factor due to the phase taper across the aperture.

$$\text{Gain (dB)} = 20\,\text{Log}\left(\frac{2\pi a}{\lambda}\right) - GF$$

The gain factor has been plotted on page 271. To find the gain of a circular horn, we must calculate S and then find the corresponding gain correction factor from the curve. This is substracted from 20 Log(circumference in wavelengths).

OPTIMUM CIRCULAR HORN

For any required gain from the horn, there is a continuum of possible designs. We can pick an S which will fix the ratio of the slant radius to the aperture radius. Given S, we find the gain correction factor from the curve. We add this to the required gain, divide by 20 and take the antilog, and we have the required aperture circumference in wavelengths. Then using S, we find the slant radius.

Example: Design a circular horn with 20 dB gain at 5 GHz.

Pick $S = .5$ From the gain correction curve we find $GF = 4.3$ dB

$$\frac{2\pi A}{\lambda} = 10^{(24.3/20)}$$

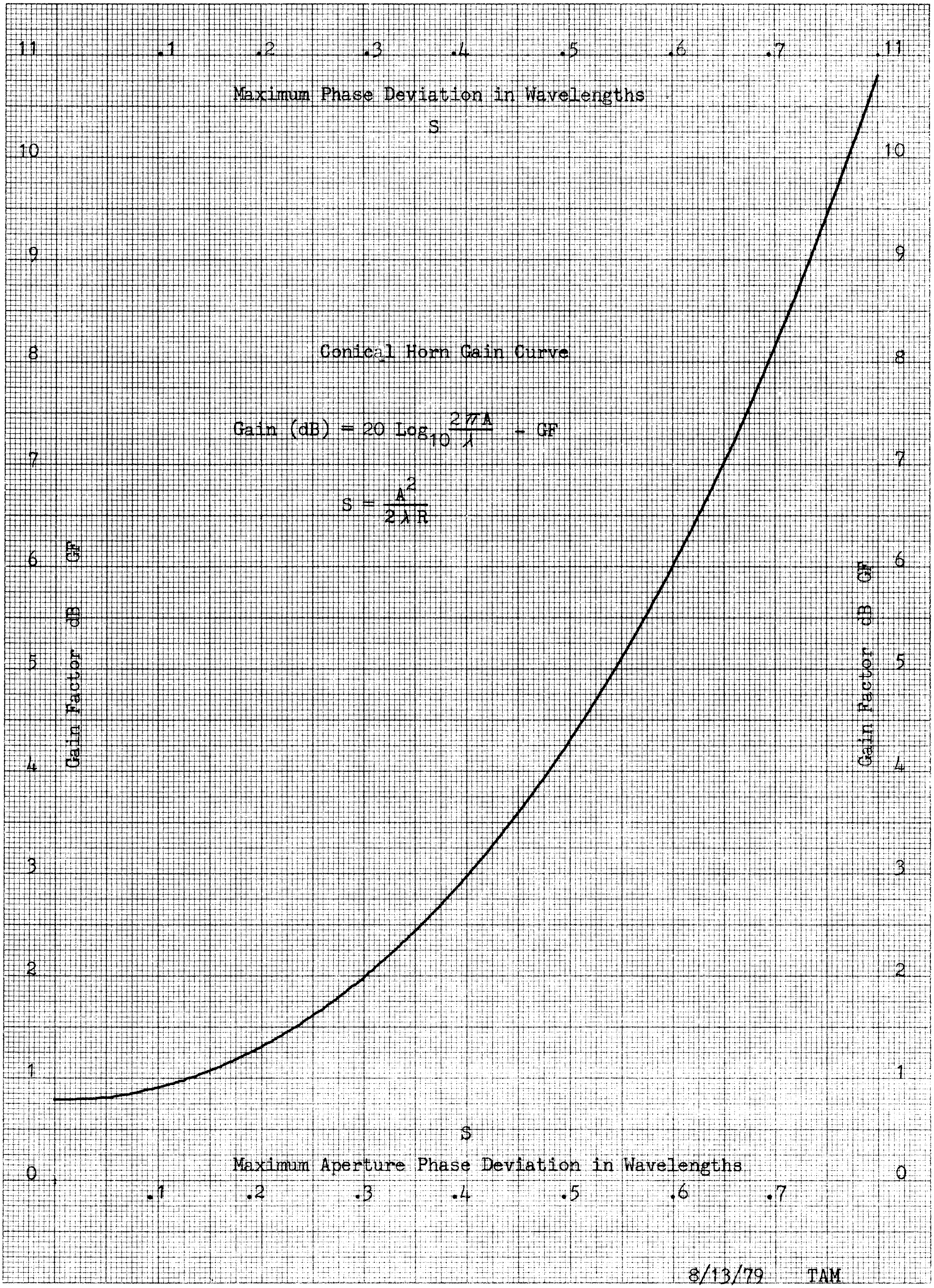
$$\frac{A}{\lambda} = 2.611$$

We have assumed that $S = A^2/(2\lambda R) = 0.5$

$$R = A^2/(2\lambda S) = (2.611\lambda)^2/\lambda$$

$$R/\lambda = 6.817$$

We have a horn with 20 dB of gain for any frequency; now we can substitute in the frequency (wavelength).


Maximum Phase Deviation in Wavelengths
S
Conical Horn Gain Curve
Gain (dB) = 20 $\log_{10} \frac{2\pi A}{\lambda}$ - GF
$S = \frac{A^2}{2\lambda R}$
Gain Factor dB GF
S
Maximum Aperture Phase Deviation in Wavelengths
.1 .2 .3 .4 .5 .6 .7
0 1 2 3 4 5 6 7 8 9 10 11
8/13/79 TAM

$$\lambda = \frac{11.80285}{5} = 2.360 \text{ inches}$$

Aperture Radius $A = 6.163$ Slant Radius $R = 16.092$

We can also pick $S = .45$ which gives a gain correction factor = 3.6 dB.

$$A/\lambda = 2.409 \qquad R/\lambda = 6.447$$
$$A = 5.687 \qquad R = 15.22$$

Both horns in the example have the same gain and meet the design goal, but it would appear that the second is better because it is smaller and cheaper to build. If we design a horn with S very small, then the aperture size will be the smallest. But R approaches infinity as S approaches zero. It appears that there is an optimum design somewhere between S = .45 and S = 0. The solution lies in plotting a curve similiar to the curves on pages 227 and 228 for the rectangular horn. On page 273 such a curve has been calculated and plotted. We find that if the voltage gain is used for the ordinate, then the points of minimum aperture for a given radial length and maximum gain line up on a single line. This is the optimum horn design line.

If we take the example and design a horn using this line we get the following design.

$$2\,A/\lambda = 4.416 \qquad R/\lambda = 6.25$$

$$A = 5.212 \qquad R = 14.753$$

This design corresponds to having $S = 0.39$.

TM_{01} MODE HORN

The TM_{01} is one of the useful overmoded circular waveguide modes. We will derive the pattern of this horn using the Huygens source approximation. From page 262 the fields in the waveguide are given by

$$E_\rho = \frac{-\chi_{01}\beta_z}{\omega\epsilon a} J_0'\left(\frac{\chi_{01}\rho}{a}\right) e^{-j\beta_z z} \qquad \chi_{01} = 2.405$$

$$E_\phi = 0$$

$$E_z = \frac{1}{j\omega\epsilon}(\beta^2 - \beta_z^2) J_0\left(\frac{\chi_{01}\rho}{a}\right) e^{-j\beta_z z}$$

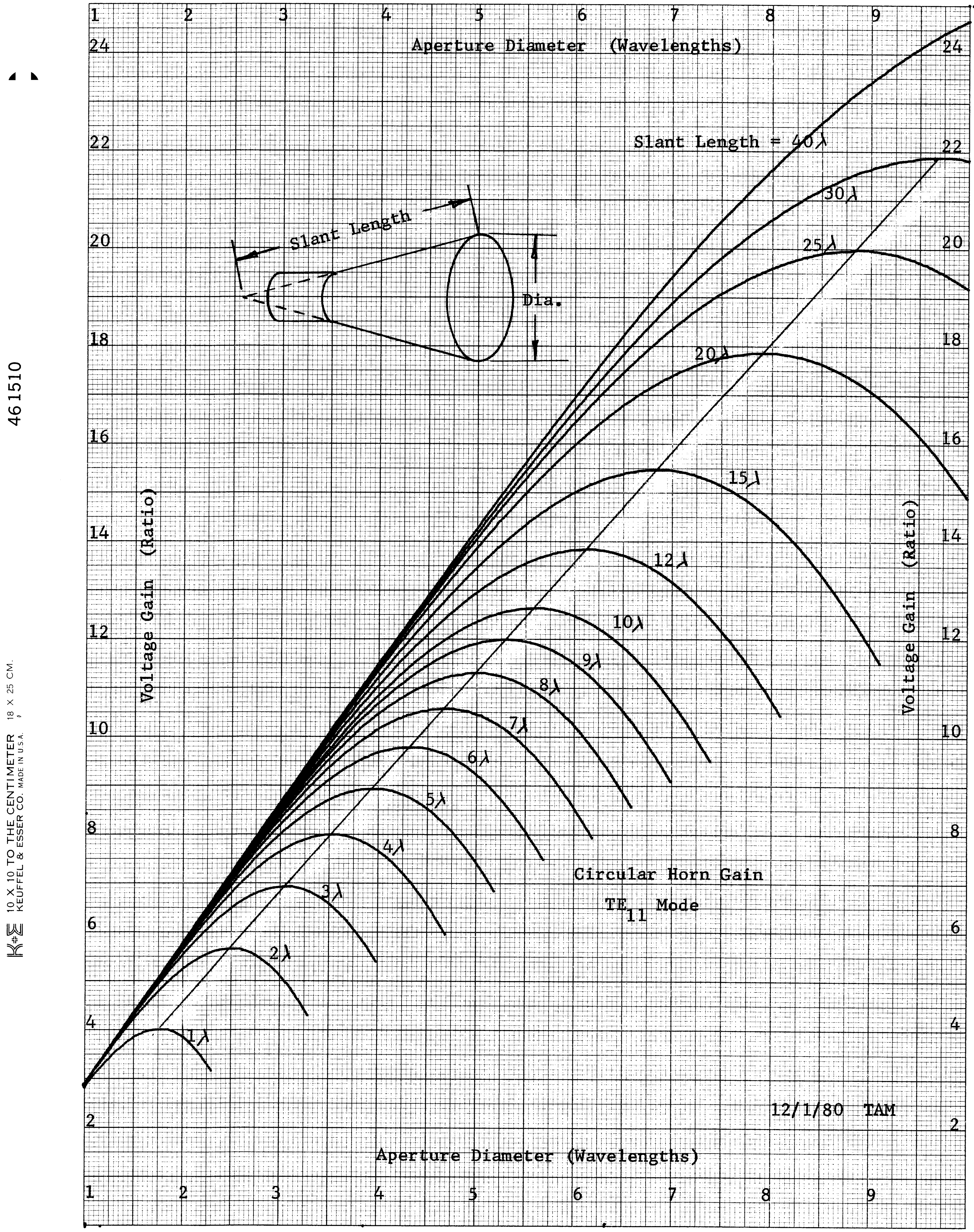
$$H_\rho = 0 \qquad H_\phi = -\frac{\chi_{01}}{a} J_0'\left(\frac{\chi_{01}\rho}{a}\right) e^{-j\beta_z z}$$

One of the properties of the Bessel function is

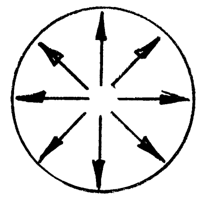
$$J_1(x) = -J_0'(x)$$

Using this, we can express the electric field in the aperture as

$$E_\rho = E_0 J_1\left(\frac{\chi_{01}\rho}{a}\right)$$


Aperture Diameter (Wavelengths)
Slant Length = 40λ
30λ
25λ
20λ
15λ
12λ
10λ
9λ
8λ
7λ
6λ
5λ
4λ
3λ
2λ
1λ
Slant Length
Dia.
Voltage Gain (Ratio)
Circular Horn Gain
TE11 Mode
12/1/80 TAM
Aperture Diameter (Wavelengths)
461510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.

The waveguide only has a radial component. The field pattern in the aperture is given in the figure below.



TM_{01} Mode

The pattern will be the same if the antenna is rotated by any amount; therefore all azimuth patterns will be the same. Let us find the θ and ϕ far field components at $\phi = 0$. The θ component is given by the following integral.

$$E_\theta = \xi E_0 \int_0^a \int_0^{2\pi} J_1\left(\frac{\chi_{01}\rho}{a}\right) \cos\phi_c \, e^{j\beta \sin\theta \cos\phi_c} \, e^{-j\frac{\pi\rho^2}{\lambda R}} \rho \, d\phi_c \, d\rho$$

The ϕ_c integral can be integrated exactly in terms of Bessel functions.

$$\int_0^{2\pi} \cos\phi_c \, e^{ju\cos\phi_c} \, d\phi_c = j\, 2\pi J_1(u)$$

This reduces the integral by completing one of the two integrals.

$$E_\theta = j\, 2\pi \, \xi E_0 \int_0^a J_1\left(\frac{\chi_{01}\rho}{a}\right) J_1(\beta\rho \sin\theta) \, e^{-j\frac{\pi\rho^2}{\lambda R}} \rho \, d\rho$$

We will reduce the integral by making the following substitutions.

$$t = \rho/a \qquad d\rho = a\,dt \qquad u = \frac{2\pi a \sin\theta}{\lambda} \qquad s = \frac{a^2}{2\lambda R}$$

These substitutions reduce the integral to a dimensionless integral.

$$E_\theta = j\, 2\pi a^2 \xi E_0 \int_0^1 J_1(\chi_{01} t) \, J_1(ut) \, t \, e^{-j 2\pi s t^2} dt$$

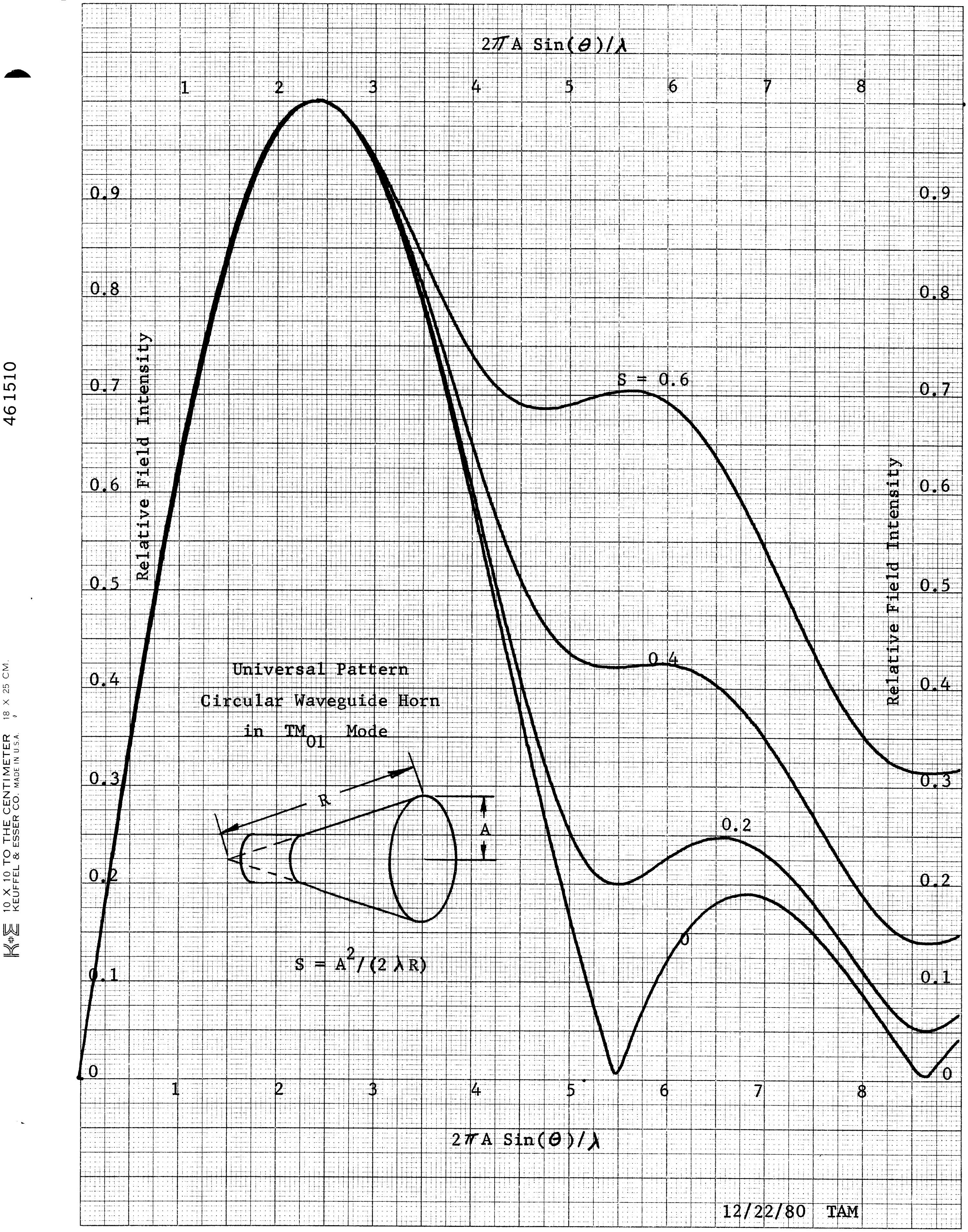
The solution of this integral is found by numerical techniques. We can use this to find a universal radiation pattern curve which is given on page 275. This is the same form as the other universal radiation curves. Because the fields in the aperture are symmetrical, all the patterns will be the same regardless of the value of ϕ. As with all universal radiation curves the ordinate is in terms of voltage and the aperture obliquity factor, $(1 + \cos\theta)/2$, has been removed from the pattern response.

The antenna will have a null on boresight. The approximate beam peak can be determined from the universal pattern. Regardless of the value of S, the peak occurs at $U = 2.43$.

$$\frac{2\pi A}{\lambda} \sin\theta = 2.43$$

$$\theta_{MAX} = \sin^{-1}\left(\frac{.387\lambda}{A}\right)$$

On page 276 is a pattern of a circular horn excited in the TM_{01} mode.


2π A Sin(θ)/λ
1
2
3
4
5
6
7
8
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Relative Field Intensity
S = 0.6
0.4
0.2
0
Universal Pattern
Circular Waveguide Horn
in TM01 Mode
R
A
S = A2/(2λR)
2π A Sin(θ)/λ
12/22/80 TAM
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10 X 10 TO THE CENTIMETER 18 X 25 CM.
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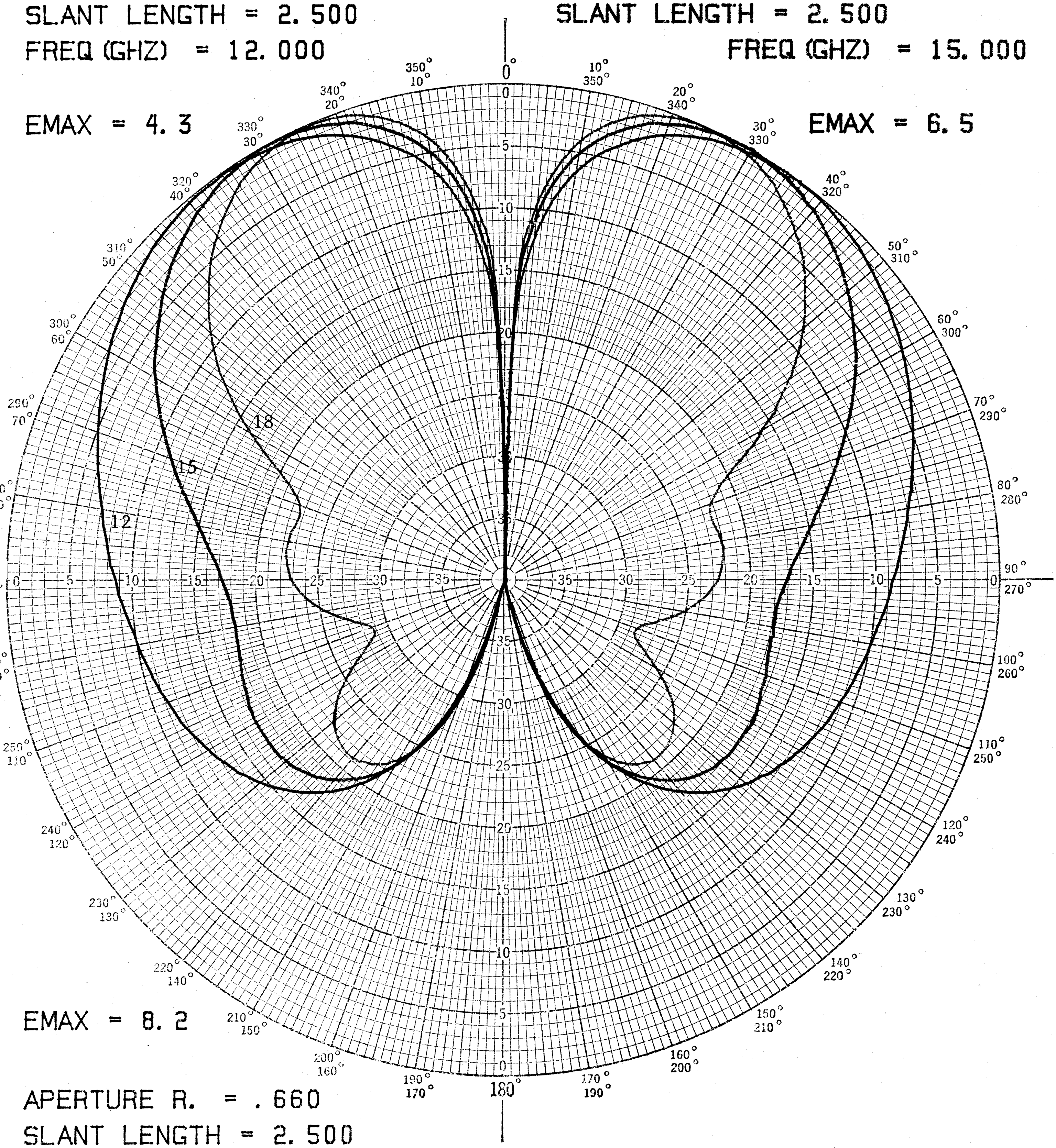
CIRCULAR WAVEGUIDE HORN IN TM-01 MODE

APERTURE R. = .660
SLANT LENGTH = 2.500
FREQ (GHZ) = 12.000

APERTURE R. = .660
SLANT LENGTH = 2.500
FREQ (GHZ) = 15.000

EMAX = 4.3

EMAX = 6.5



EMAX = 8.2

APERTURE R. = .660
SLANT LENGTH = 2.500
FREQ (GHZ) = 18.000

Polar Chart No. 127D
SCIENTIFIC-ATLANTA, INC.
ATLANTA, GEORGIA

Now let us consider the ϕ component. From symmetry alone we can see that the ϕ component is zero. The fields are equal and opposite in direction across a line symmetry drawn radially along $\phi_c = 0$. These fields will cancel each other. We can also find this from the mathematics. The ϕ component is found from this integral at $\phi = 0$.

$$E_\phi = \xi E_o \int_0^{2\pi}\int_0^{a} J_1\left(\frac{\chi_{01}\rho}{a}\right) \sin\phi_c \, e^{j\beta\rho \sin\theta\cos\phi_c} e^{-j\frac{\pi\rho^2}{\lambda R}} \rho \, d\rho \, d\phi_c$$

When we integrate the ϕ_c component, we find

$$\int_0^{2\pi} \sin\phi_c \, e^{ja\cos\phi_c} \, d\phi_c = 0$$

From symmetry all integrals for an arbitrary ϕ are zero and there is no ϕ component in the far field. This horn is similiar to the TM_{11} mode in rectangular waveguide. When mounted on a model tower positioner, the antenna is always horizontally polarized. If mounted on a satellite and pointed toward the earth, it would receive signals which are vertically polarized on the earth except straight down where there is a null in the pattern.

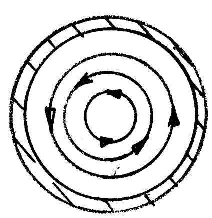
TE_{01} MODE HORN

The TE_{01} mode horn is the dual of the TM_{01} mode horn. This horn is always vertically polarized when mounted on a model tower positioner. From the equations on page 261 we find the waveguide fields as

$$E_\phi = \frac{\chi'_{01}}{a} J_0'\left(\frac{\chi'_{01}\rho}{a}\right) e^{-j\beta_z z} \qquad \chi'_{01} = 3.832$$

$$E_\phi = E_o J_1\left(\frac{\chi'_{01}\rho}{a}\right)$$

The second expression is the equation of the field in the aperture plane without the quadratic phase factor and X'_{01} is the first zero of the derivative of $J_0(x)$. These fields are given in the figure below.



TE_{01} Mode

Similiar to the TM_{01} mode, the fields are the same in the aperture if the antenna is rotated by any amount and all azimuth patterns will be the same. The far field ϕ component is found from the following integral.

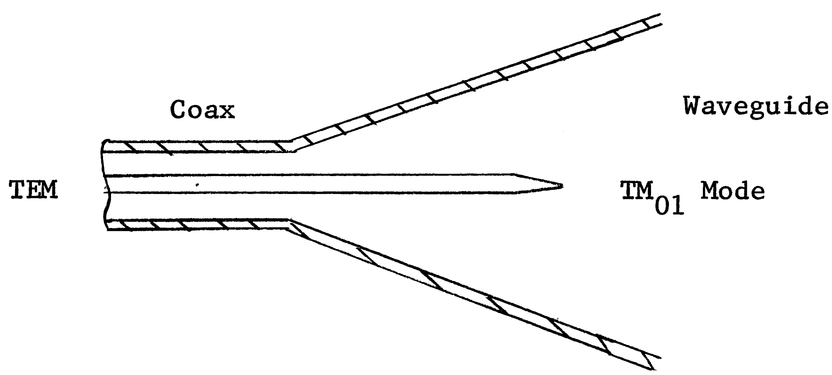
$$E_\phi = \xi E_o \int_0^{2\pi}\int_0^{a} J_1\left(\frac{\chi'_{01}\rho}{a}\right) \cos\phi_c \, e^{j\beta\rho \sin\theta\cos\phi_c} e^{-j\frac{\pi\rho^2}{\lambda R}} \rho \, d\rho \, d\phi_c$$

This is the same integral as the TM_{01} pattern integral except for a constant in the Bessel function. We can perform similiar integrations and substitutions on the integral as were done on the TM_{01} integral and find a universal radiation pattern which is given on page 279. It is similiar to the TM_{01} mode pattern. The ordinate is a voltage pattern and the obliquity factor, $(1 + \cos \theta)/2$, has beeen removed from the pattern. The maximum of the pattern occurs at approximately U = 2.9 for small values of S (large slant radius, R). Using this the maximum of the pattern is found.

$$\theta_{MAX} = \sin^{-1}(.462\ \lambda/A) \quad \text{for reasonable tapers}$$

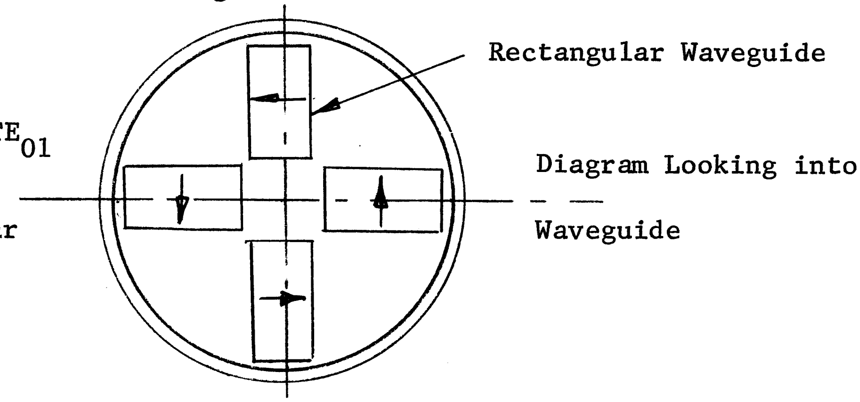
EXCITATION OF HIGHER ORDER CIRCULAR WAVEGUIDE MODES

The TM_{01} mode can be excited directly from coax by flaring the outer shield of a coax until the diameter is sufficient to support the TM_{01} mode. At that point the center conductor can be tapered to nothing and the TM_{01} mode is launched in the waveguide. If good symmetry is maintained, then the TE_{11} lower order mode will not be excited.

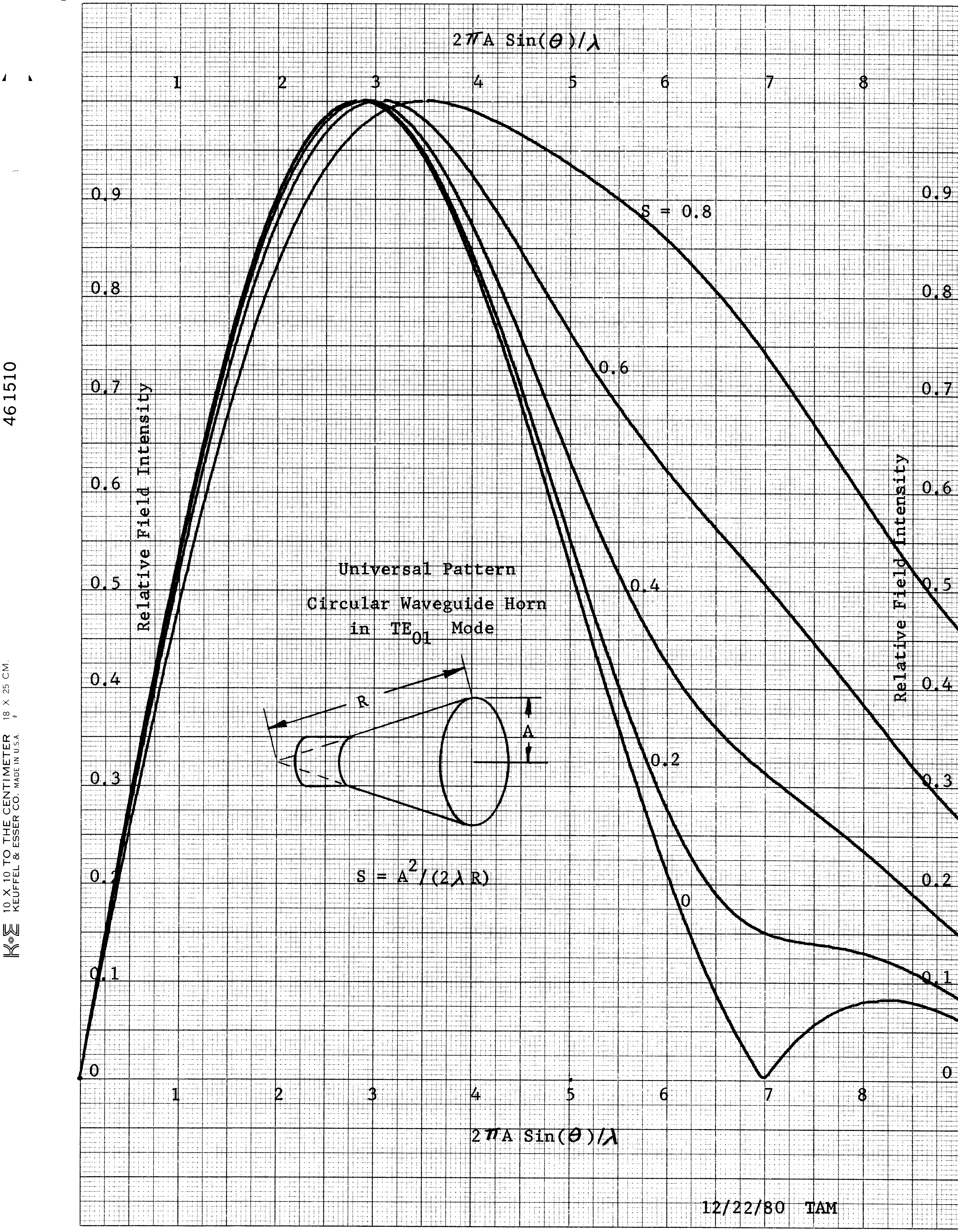


This is a handy way to start the waveguide horn because the center conductor of the coax can be tapered to give a broadband match between the coax and the horn. When a waveguide alone is being excited, the transition between the coax and the waveguide is made abruptly and the center pin of the coax becomes a probe which excites the waveguide. The length of the probe is adjusted to give a good match.

The TE_{01} mode is more difficult to excite cleanly. It is the fourth possible mode in the waveguide and so the other modes will be excited easily. This mode will be excited when the end of the waveguide is fed by four rectangular waveguides arranged as in the figure below.



Excitation of TE_{01} Mode in Circular Waveguide


2πA Sin(θ)/λ
1
2
3
4
5
6
7
8
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Relative Field Intensity
S = 0.8
0.6
0.4
0.2
0
Universal Pattern
Circular Waveguide Horn
in TE01 Mode
R
A
S = A2/(2λR)
12/22/80 TAM
461510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.

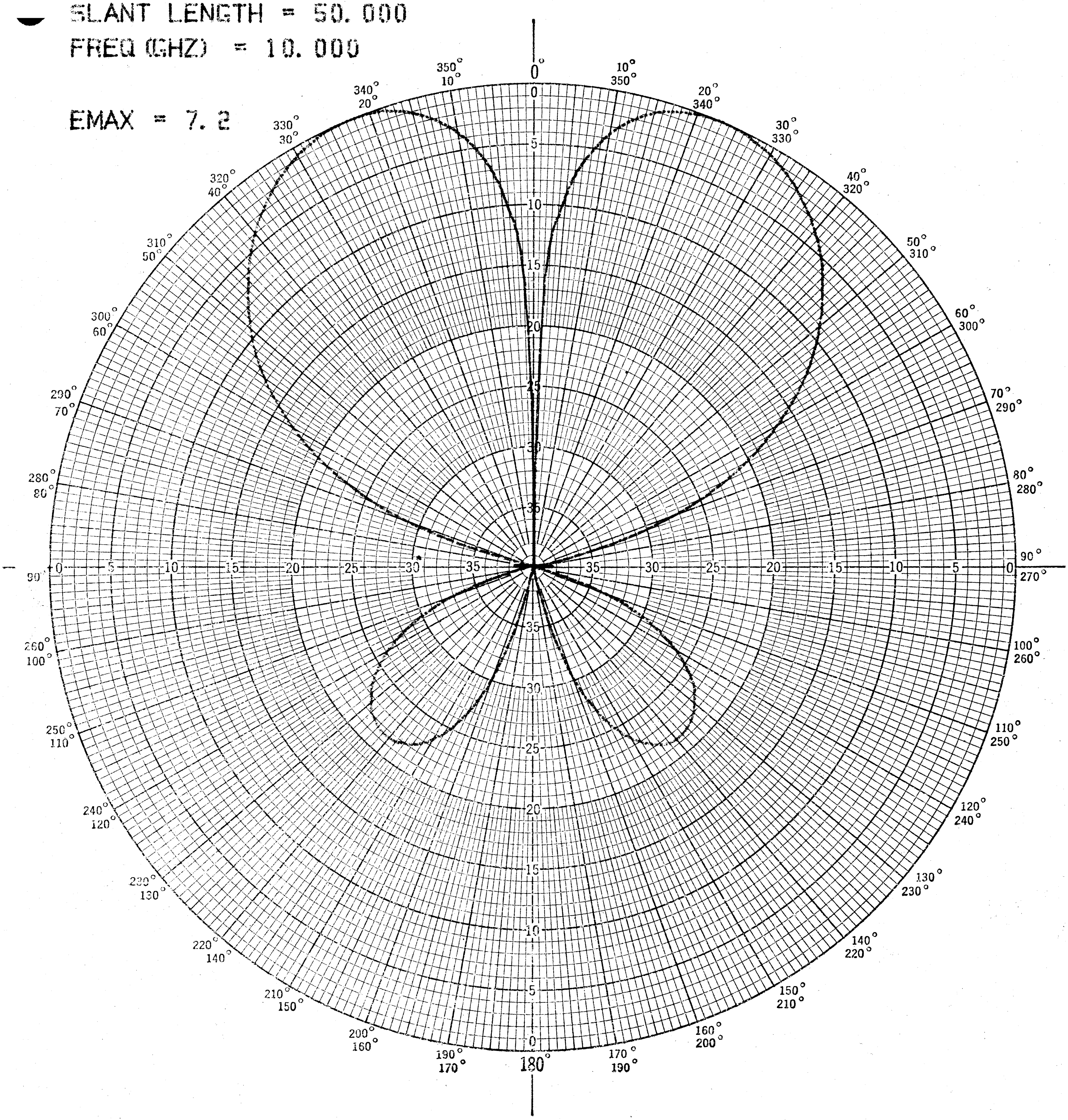
CIRCULAR WAVEGUIDE HORN IN TE-01 MODE

APERTURE R. = 1.350

SLANT LENGTH = 50.000

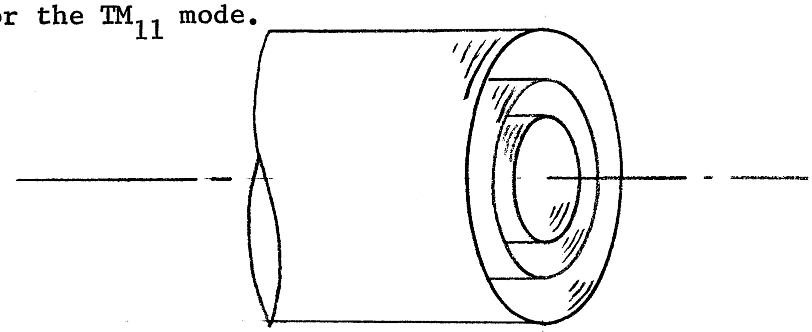
FREQ (GHZ) = 10.000

EMAX = 7.2



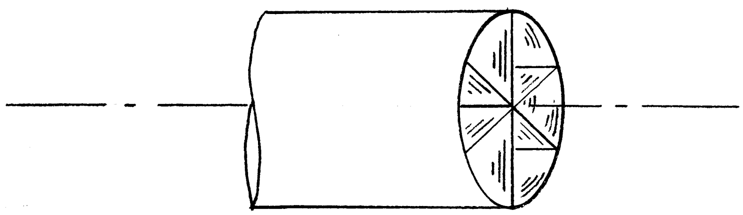
If the rectangular waveguides are fed in phase as shown in the diagram, then the TE_{01} modes will be excited in the circular waveguide.

One method of cleaning up the modes is to use mode suppressors in the waveguide to cut off all unwanted modes. The following is used as a mode suppressor for the TM_{11} mode.



The TM_{01} mode will be unaffected by a series of concentric cylinders in the waveguide. These will suppress the TE_{11} mode.

An unwanted mode suppressor that is used with the TE_{01} mode is shown below.



The radial fins in the waveguide can be used to suppress the TE_{11}, TM_{01}, and TE_{21} modes in the circular waveguide. One problem with this method is that when the mode suppressor ends, the discontinuity will regenerate the lower order modes again. The suppressor cannot completely eliminate the lower order modes.

SLOTS IN CIRCULAR WAVEGUIDES

Before we leave the discussion of circular waveguides, we need to discuss cutting slots in the waveguide walls. We will assume that the slot is a resonant length and very thin. The electric field will be across the slot regardless of the fields or currents exciting it. The slot is excited when it cuts currents in the waveguide walls.

We must first calculate the currents in the walls of the waveguide. The currents can be found from the tangential magnetic fields and application of the boundary conditions given on page 154; the surface current density is given by:

$$\bar{J}_s = \hat{n} \times \bar{H}$$

The normal vector, $\hat{n}$, to the walls is $-\bar{a}_\rho$. Using this we can expand the curl by the determinant notation.

$$J_s = -\begin{vmatrix} \bar{a}_\rho & \bar{a}_{\phi_c} & \bar{a}_z \\ 1 & 0 & 0 \\ H_\rho & H_{\phi_c} & H_z \end{vmatrix} = H_z \bar{a}_\phi - H_\phi \bar{a}_z$$

TE_{11} MODE CURRENTS

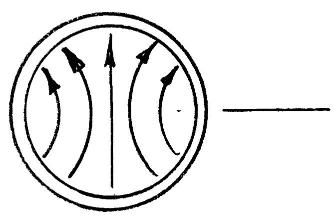
Using the equations for the TE modes on page 261, we find both components of the current on the wall.

$$J_\phi = \frac{E_0}{j\omega\mu}(\beta^2 - \beta_z^2)\, J_1(\chi_{11}')\cos\phi\, e^{-j\beta_z z}$$

$$J_z = \frac{\beta_z E_0}{\omega\mu a} J_1(\chi_{11}')\sin\phi\, e^{-j\beta_z z}$$

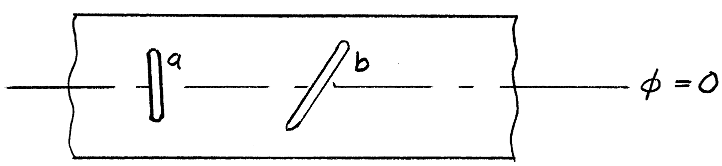
Notice that the Z directed currents are in phase with the electric field and are traveling waves in the guide. The ϕ directed currents are 90 degrees out of phase with the electric field and are standing wave currents. We need to redraw the mode pattern in the waveguide to establish the coordinate system.

TE_{11} Mode



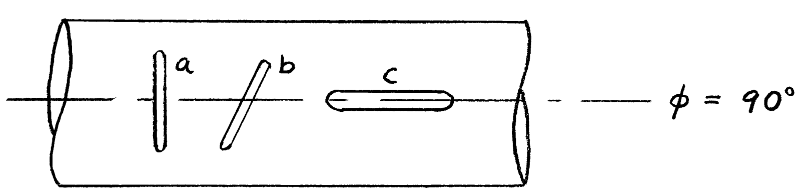
$\phi = 0$

Let us consider the wall at $\phi = 0$. The Z directed currents are zero (Sin 0 = 0). In the figure below, the slot a does not cut any net current. The current below the line $\phi = 0$ is matched by current above $\phi = 0$ which is flowing in the opposite direction (Z directed currents).



The slot b will be excited by ϕ directed currents which are at a peak (Cos 0 = 1). These slots, centered at $\phi = 0$, are the same as sidewall slots in rectangular waveguide. Since they are excited by the circulating ϕ currents, they are shunt loads to the waveguide.

Consider the slots centered about $\phi = 90^\circ$.



The slot a will cut currents in the Z direction and will be excited. The slot b is also excited by Z directed currents; the ϕ directed currents on both side of the center line that we cut by the slot are equal and opposite. The slot c is not excited because it can only cut ϕ directed currents which are zero at ϕ = 90 degrees ($\cos(90^{\circ}) = 0$). This slot would be used for probing the waveguide to measure VSWR. These slots are similiar to the top wall slots of the rectangular waveguide. Since they cut the traveling wave currents, they are series loads to the waveguide.

All slots which are placed so that they are not centered on these two axes will be excited. Slots with their long axis parallel to the Z axis will be shunt loads to the waveguide because they cut standing wave currents. Slots with their long axis in the ϕ direction will be series loads to the waveguide because they cut traveling wave currents. All slots which are not parallel to these axes will be compound loads to the waveguide.

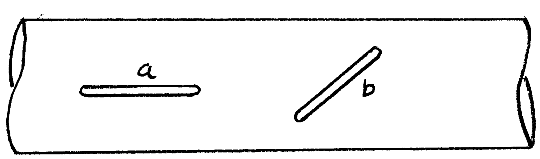
TM_{01} MODE CURRENTS

We can use the equations on page 262 to find the wall currents in the circular waveguide when propagating the TM_{01} mode.

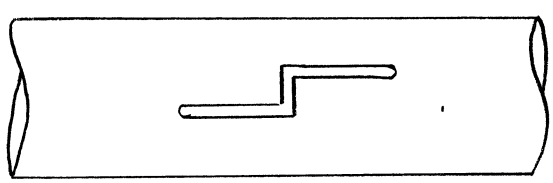
$$J_\phi = 0 \quad \text{Since} \quad H_z = 0 \quad \text{(True of all TM modes)}$$

$$J_z = -H_\phi = \frac{X_{01}}{a} J_o'(X_{01})\, e^{-j\beta_z Z}$$

The currents in the side walls are only Z directed and there is no ϕ dependence. The currents are the same as those for a coax line in the TEM mode.



The slot a is not excited, but b which cuts Z directed currents is excit-ed. Another method of exciting slots is to stagger cut the slot in the wall as shown below.



This slot, whose total length is resonant, is excited by the small portion of the slot cutting Z directed currents. The excitation level can be varied by changing this length.

Notice that there is a current flowing on the wall in the Z direction but there is no return current as in the case of TEM mode coax or TE_{11} mode circular waveguide.

TE_{01} MODE CURRENTS

The wall currents of the TE_{01} mode are found to be

$$J_\phi = \frac{E_0}{j\omega\mu}(\beta^2 - \beta_z^2)\, J_0(\chi_{01})\, e^{-j\beta_z z}$$

but X_{01} is the first zero of the Bessel function $J_0(x)$. There are no ϕ directed currents. Since $H_\phi = 0$ $(n = 0)$, there are no Z directed currents either. There are no wall currents at all for the TE_{01} mode. Any slots that are cut in the walls of the waveguide supporting the TE_{01} mode will not be excited.

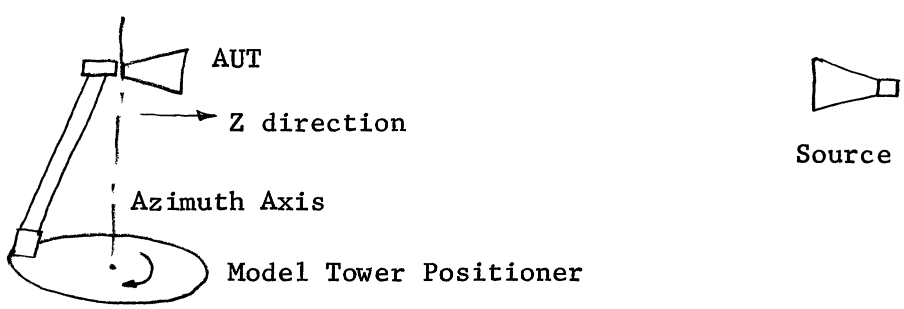
Most of the losses in waveguide are due to wall currents, because there are no wall currents in the TE_{01} mode, the losses are very small. Many years of work have been devoted to investigating this mode for use in transmitting telephone channels. Bends are a big problem because they generate lower order modes.

CORRUGATED HORNS

One of the chief uses of horns is for feeds of large reflector antennas. In these applications it is desirable or required to have dual linear polarization. In other applications a square or circular aperture horn is fed with cross polarized linear feeds in quadrature (90° out of phase) to obtain circular polarization. One of the problems with the square or circular aperture antenna is that it has different E and H plane beamwidths. We will not get optimum feed efficiency with unequal beamwidths. When used as a circularily polarized antenna, the off axis axial ratio will be degraded because there is a different response to the two linear polarizations. The second problem with circular or square aperture horns is a phase center variation when used as a feed for a parabolic reflector.

Phase Center

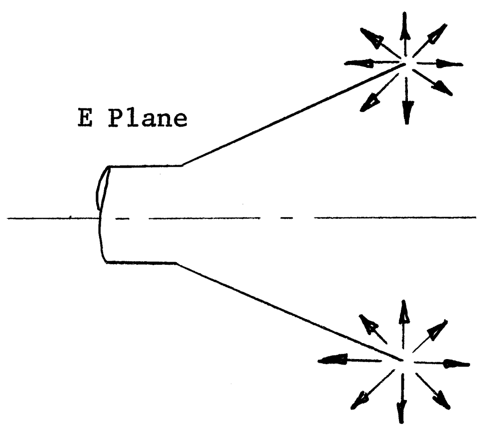
The phase center of an antenna is that point on the antenna from which it appears the radiation is coming from if it were a point source. The antenna pattern has a phase as well as amplitude pattern. Practically the phase is measured with respect to some reference signal which could come from a reference horn placed in the field of the source and not moved or it could be a portion of the signal coupled off at the signal generator. We are only interested in the relative phase of one point on the pattern to another, so we usually reference the phase to some part on the pattern such as boresight. Remember the true phase is due to the signal traveling from the source to the antenna under test $(e^{-j\beta d})$, where d is many wavelengths.



If we mount the antenna under test (AUT) on a model tower positioner and rotate the azimuth axis around the antenna boresight, we will find a position of the Z axis which will give us a flat phase response. The point where the azimuth axis interects the antenna will be the phase center. This is the point where the antenna appears to transmit spherical waves from. The antenna may also have lateral offset where the axis of the phase center of the antenna under test does not intersect the azimuth axis. In that case the antenna will not have equal phases at equal angles on both sides of $\theta = 0$. Lateral offset is usually a problem with the measuring equipment or faulty antenna construction or mounting on the positioner.

If we take a horn and measure its phase center, then we will find it somewhere along the flare of the horn. When we measure a square or circular aperture horn, we find that the phase center location in the E and H planes are located in different places. A feed for a parabolic reflector should have its phase center at the focus of the parabola. When it is not there, the feed/reflector system will have phase error loss.

A third problem with horns is a limited front to back ratio. The analysis used so far tells us nothing about the backlobe (only near boresight). The backlobe energy is due to signals which are diffracted off the E plane edges of the horn.



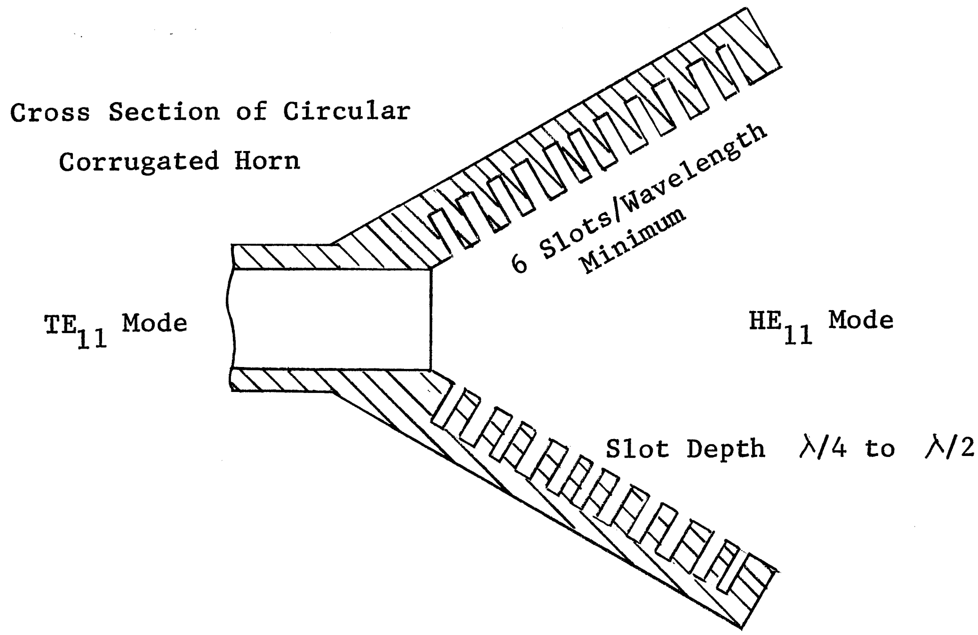
This is similiar to the diffraction off the edges of ground planes discussed on page 190. Each edge acts as a secondary source of radiation. The diffracted energy can be reduced if the aperture field is reduced on these edges.

The corrugated horn is a solution to all these problems. We will discuss only the circular corrugated horn. Similiar results hold for the square corrugated horn, but it has the disadvantage that not all azimuth patterns will be the same. Those patterns through the diagonals have narrower beamwidths than those through the E and H planes.

The corrugations are cut in the walls and extend all around the horn in the ϕ direction. The corrugations must satisfy these conditions.

1. The depth of the slots must be between $\lambda/4$ and $\lambda/2$.

2. There must be at least 6 slots per wavelength along the slant radius.



The first condition prevents having surface waves develop on the corrugations. We can look at the corrugations as parallel plate transmission lines. These shorted lines will give a reactive boundary at the junction of the slots and the horn flare. The boundary will support surface waves when it is inductive. It is inductive for a depth less than $\lambda/4$ or greater than $\lambda/2$. For a circular waveguide we should represent the corrugations as radial waveguides where the fields are described by Bessel functions. The solution for the fields in the waveguide is found by matching the radially directed impedances at the boundaries. The solution in the waveguide is in terms of Bessel functions. When the matching is performed, we get an approximate field in the corrugated waveguide[1].

$$E_\rho = J_0\left(\frac{\chi_{01}\rho}{a}\right)\cos\phi_c$$

$$E_\phi = -J_0\left(\frac{\chi_{01}\rho}{a}\right)\sin\phi_c$$

χ_{01} is the first zero of $J_0(x)$

This is called the HE_{11} mode which is a hybrid mode. It is a combination of the TE_{11} and TM_{11} modes. We will use this field to calculate the properties of the circular corrugated horn by the Huygens source method. This field gives equal E and H plane patterns and the field is zero at the corrugations so that the edge diffraction is eliminated. Because the amplitude distribution is the same radially for every ϕ, all azimuth patterns are the same. The only phase distribution across the aperture is due to the flare angle of the horn. Since we have both amplitude and phase symmetry across the aperture, each azimuth pattern will have the same phase center and there is a uniquely defined phase center.

The integral for the far field pattern is

$$E_\theta = \xi\int_0^{2\pi}\int_0^{a} J_0\left(\frac{\chi_{01}\rho}{a}\right) e^{j\beta\rho\sin\theta\cos\phi_c}\, e^{-j\frac{\pi\rho^2}{\lambda R}}\,\rho\, d\rho\, d\phi_c$$

1. P. J. B. Clarricoats and P. K. Saha, "Propagation and Radiation of Corrugated Feeds", Proc. IEE, vol. 118, pp. 1167-1176, Sept. 71

We can solve the ϕ integration by using a Bessel function result.

$$\int_0^{2\pi} e^{j\, u \cos \phi_c}\, d\phi_c = 2\pi\, J_0(u)$$

The integral for all azimuth cuts (any ϕ) becomes

$$E = 2\pi\, \xi \int_0^{a} J_0\left(\frac{\chi_{01}\rho}{a}\right) J_0(\beta\rho \sin\theta)\, e^{-j\frac{\pi\rho^2}{\lambda R}}\, \rho\, d\rho$$

We can reduce this to a dimensionless integral by making some substitutions.

$$t = \rho/a \qquad u = \frac{2\pi a \sin\theta}{\lambda}$$

$$E = 2\pi a^2\, \xi \int_0^{1} J_0(\chi_{01} t)\, J_0(ut)\, e^{-j2\pi s t^2}\, t\, dt$$

The integral can be used to draw a universal radiation pattern given on page 288. This pattern is similiar to the H plane pattern of the circular waveguide horn. The pattern as usual does not include the obliquity factor, $(1 + \cos\theta)/2$, and the ordinate is the voltage pattern (3 dB - 0.707).

The gain may be found by finding the total power in the aperture.

$$P = \int_0^{2\pi}\int_0^{a} J_0^2\left(\frac{\chi_{01}\rho}{a}\right)\rho\, d\rho\, d\phi$$

This is reduced to a dimensionless integral by the substitution: $t = \rho/a$

$$P = 2\pi a^2 \int_0^{1} J_0^2(\chi_{01} t)\, t\, dt$$

The directivity is found from the formula: $\quad \text{Directivity} = \dfrac{4\pi\, |E_{MAX}|^2}{P}$

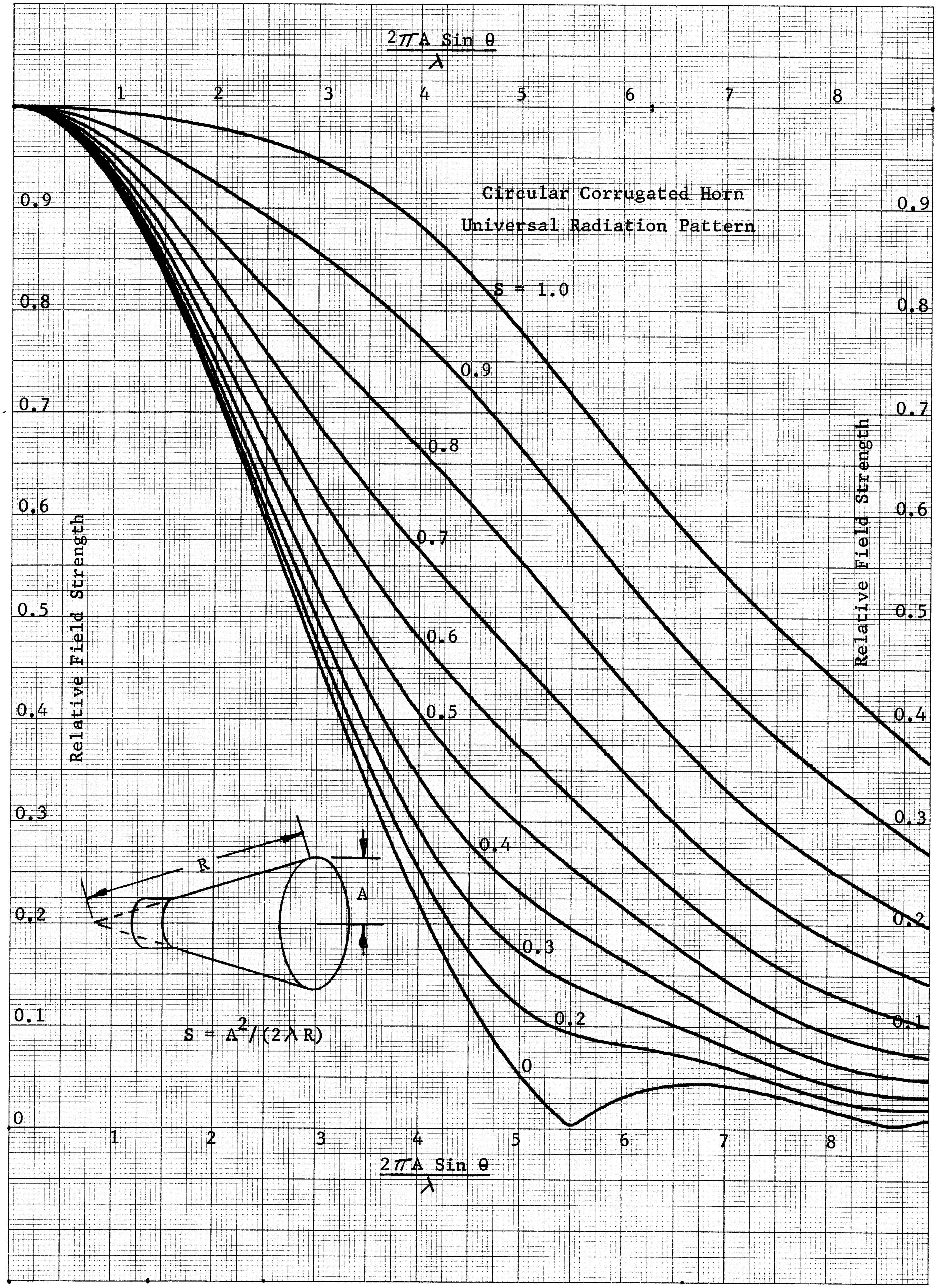
$$\text{Directivity} = \left(\frac{2\pi a}{\lambda}\right)^2 \left[\frac{2\left|\int_0^1 J_0(\chi_{01} t)\, e^{-j2\pi s t^2}\, t\, dt\right|^2}{\int_0^1 J_0^2(\chi_{01} t)\, t\, dt}\right]$$

The term in the brackets can be identified as a gain correction factor which is only a function of S, the maximum aperture phase deviation in wavelengths. The gain (directivity) is then given by

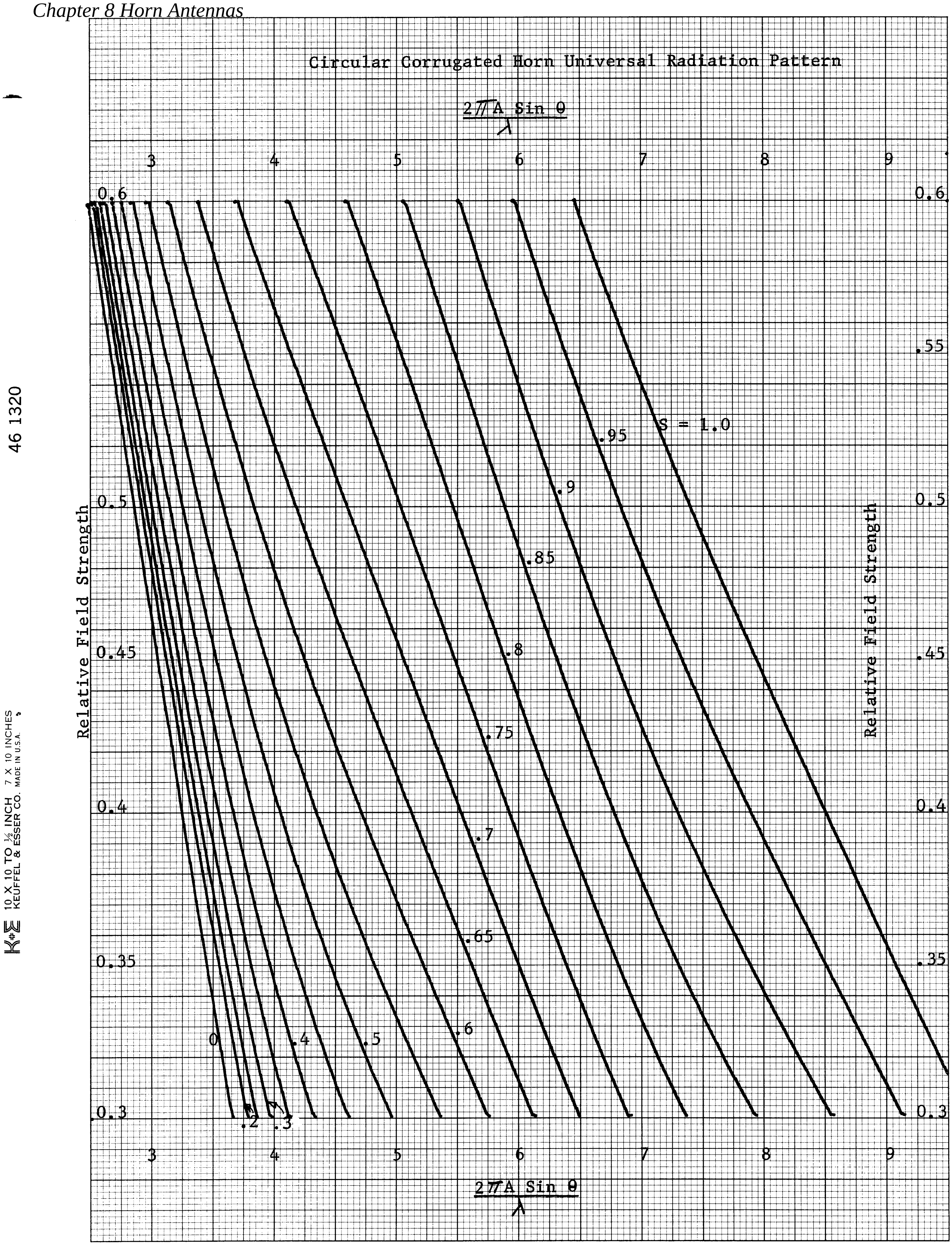
$$\text{Gain (dB)} = 20 \operatorname{Log}\left(\frac{2\pi a}{\lambda}\right) - GF$$

The gain correction factor is plotted on page 289. We can use this curve to design a horn given the required gain. Suppose we want a horn with 20 dB gain at 6 GHz.

Pick S = .4, from the curve we find the gain correction factor 3.02 dB.



$\frac{2\pi A \sin\theta}{\lambda}$
1 2 3 4 5 6 7 8
Circular Corrugated Horn
Universal Radiation Pattern
S = 1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0
Relative Field Strength
R
A
$S = A^2/(2\lambda R)$



Circular Corrugated Horn Universal Radiation Pattern
2π A Sin θ / λ
3
4
5
6
7
8
9
0.6
.55
0.5
0.45
.45
0.4
0.35
.35
0.3
Relative Field Strength
Relative Field Strength
S = 1.0
.95
.9
.85
.8
.75
.7
.65
.6
.5
.4
.3
.2
0
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

The aperture size is found from

$$\frac{A}{\lambda} = \frac{1}{2\pi}\, 10^{(20 + 3.02)/20} = 2.25$$

The slant length or radius is determined by the choice of S = .4 .

$$\frac{R}{\lambda} = \frac{(A/\lambda)^2}{2\,S} = 6.347$$

When we substitute in the frequency (wavelength), the final dimensions are:

Aperture Radius, A = 4.441 inches

Slant Radius, R = 12.53 inches

We can find out if there is an optimum design by drawing a curve similiar to the curve on page 273 for the circular waveguide horn. This curve is drawn on page 291. We can see that there is an optimum gain line for this horn. This line corresponds to S = .484 approximately, where the maximum gain is obtained for the minimum slant radius. The optimum design for the example above is found by using S = .484 . The gain correction factor is 3.71.

$$\frac{A}{\lambda} = \frac{1}{2\pi}\, 10^{(20 + 3.71)/20} = 2.44$$

The slant radius is determined by the choice S = .484

$$\frac{R}{\lambda} = \frac{(A/\lambda)^2}{2\,S} = 6.152$$

The dimensions for the frequency = 6 GHz are aperture radius = 4.8 inches and slant radius = 12.102 inches. The aperture radius has increased over the previous design but the slant radius is a minimum.

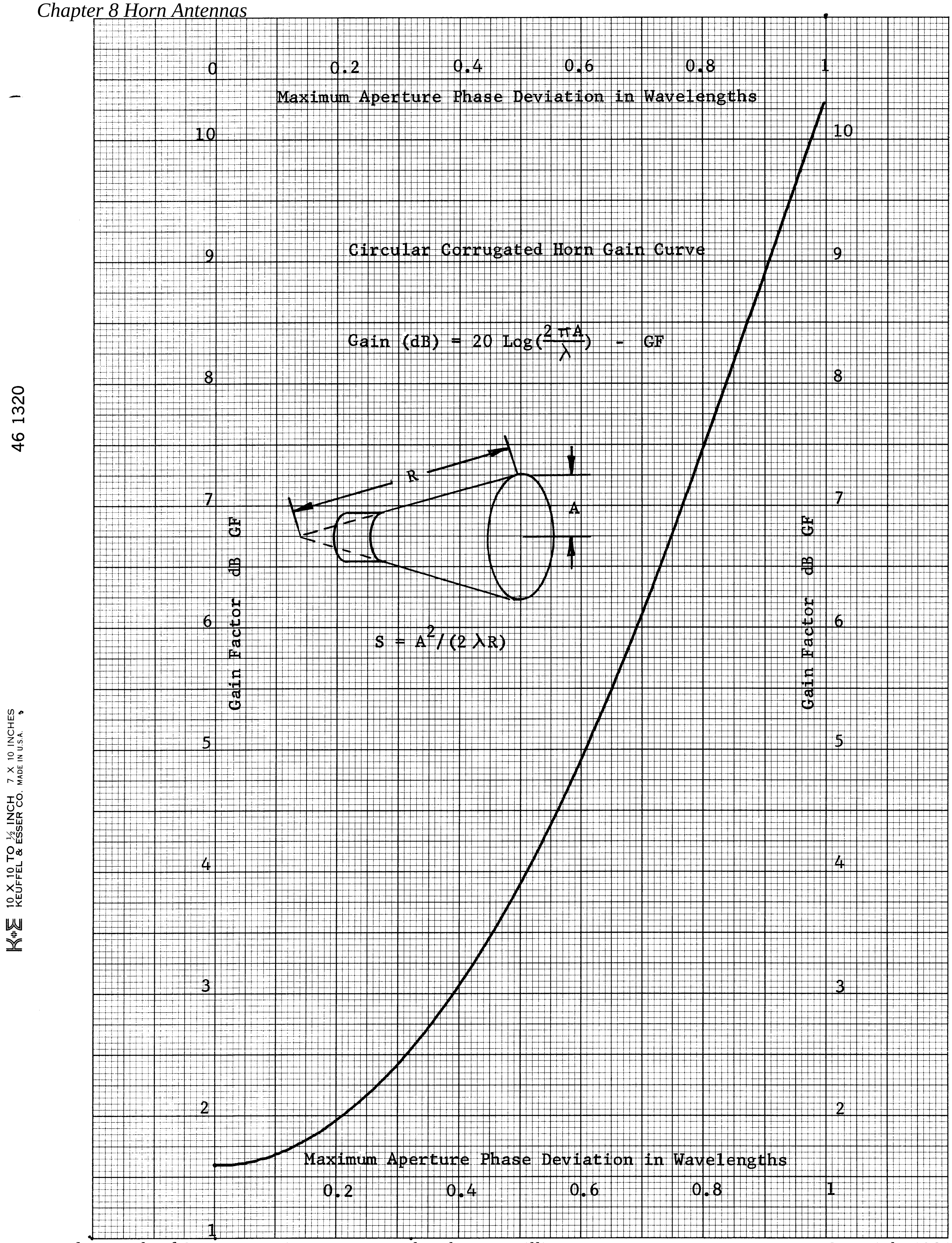
Since the horn is mostly used for feeding a reflector, we are interested in designing to a particular beamwidth. The beamwidth of most interest is the 10 dB beamwidth because this is the level at the edge of the dish which gives good pattern response and efficiency of the feed. On page 292 is a blown up portion of the curve on page 288 around some common specified gain levels.

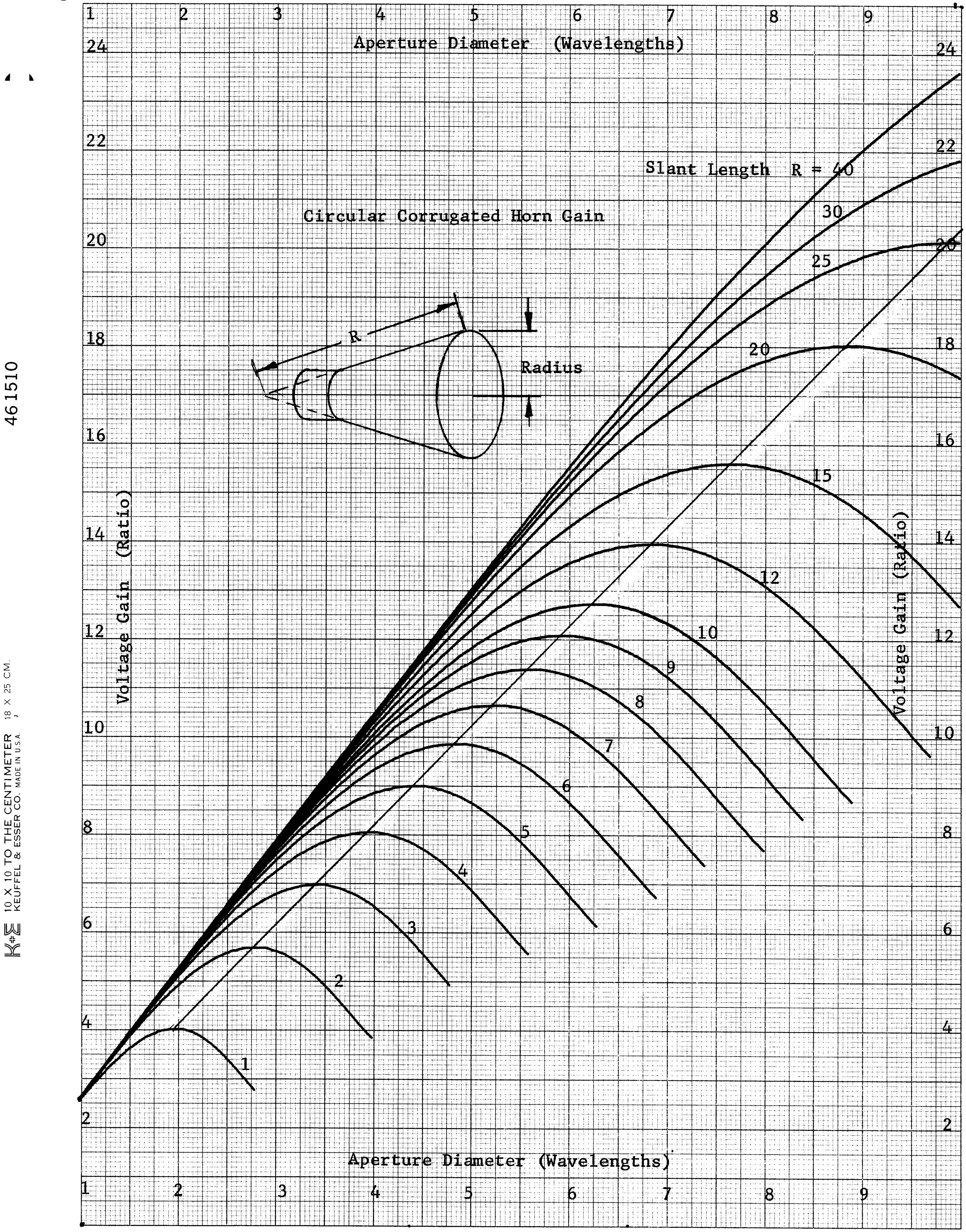
Example: Design a corrugated circular horn with 100^o, 10 dB beamwidth.

We use the curve on page 292 to design this. Pick S = .2, this will establish the relation between the aperture radius and the slant radius. Since we know the level of the pattern we want, we can find the obliquity factor: $(1 + \mathrm{Cos}(50^o))/2 = .821$

The point of the graph we are interested in is $\dfrac{10^{-10/20}}{.821} = .385$

This is a horizontal line on the curve on page 292. At the intersection of the line and the curve S = .2 we find the value U = 3.20


Maximum Aperture Phase Deviation in Wavelengths
0
0.2
0.4
0.6
0.8
1
Circular Corrugated Horn Gain Curve
Gain (dB) = 20 Log(2πA/λ) - GF
R
A
S = A²/(2λR)
Gain Factor dB GF
Gain Factor dB GF
1
2
3
4
5
6
7
8
9
10
Maximum Aperture Phase Deviation in Wavelengths
0.2
0.4
0.6
0.8
1
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


Circular Corrugated Horn Gain
Aperture Diameter (Wavelengths)
Voltage Gain (Ratio)
Slant Length R = 40
30
25
20
15
12
10
9
8
7
6
5
4
3
2
1
R
Radius
461510
10 X 10 TO THE CENTIMETER 18 X 25 CM.
KEUFFEL & ESSER CO. MADE IN U.S.A.

$$U = \frac{2\pi A \sin\theta}{\lambda}$$

$$\frac{A}{\lambda} = \frac{3.20}{2\pi \sin(50)} = 0.665$$

$$\frac{R}{\lambda} = \frac{(A/\lambda)^2}{2\,S} = 1.105$$

When the frequency is supplied, the wavelength is calculated and the final dimensions can be found.

It is not always necessary to have corrugations over the whole flare of the horn. The number of corrugations is found by building a horn with corrugations throughout the flare of the horn and experimenting to find the minimum required by taping over the corrugations with metal tape. Most people like to start with all the corrugations taped over so that the improvement can be measured with the corrugations. The corrugations are expensive to machine so the minimum number is the best design. Many of these horns are made by electroforming the horn. This requires that a mandel be made out of aluminum that has the shape of inside of the horn. On this mandel copper is deposited by a plating process. When the thickness of the copper is sufficient, the aluminum mandel is dissolved out of the horn. Of course, a mandel must be made for each horn, but the machining is simpler on the mandel than inside a horn.

The phase center of the horn is usually best found experimentally. It has been reported to be near the throat of the horn and that its variation with frequency is small. The location of the phase center is important when it is used as a feed for a reflector antenna.

RECTANGULAR CORRUGATED HORN

The rectangular corrugated horn will approximately have a cosine distribution in both planes. This is the H plane distribution of a rectangular horn whose universal radiation pattern is plotted on page 221. We can use this plot to design antennas with a given beamwidth. Both principle plane cuts will be the same as H plane cuts.

The gain can be found from the analysis of the plain rectangular horn. When Schelkunoff's equations are rearranged, a gain formula is obtained.

$$\text{Gain} = \frac{4\,R_x\,R_y\,\pi}{a\,b}\left((C(u_x) - C(v_x))^2 + (S(u_x) - S(v_x))^2\right)\left((C(u_y) - C(v_y))^2 + (S(u_y) - S(v_y))^2\right)$$

where

$$u_x = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_x}}{a} + \frac{a}{\sqrt{\lambda R_x}}\right) \qquad v_x = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_x}}{a} - \frac{a}{\sqrt{\lambda R_x}}\right)$$

$$u_y = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_y}}{b} + \frac{b}{\sqrt{\lambda R_y}}\right) \qquad v_y = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R_y}}{b} - \frac{b}{\sqrt{\lambda R_y}}\right)$$

$C(u)$ and $S(u)$ are the cosine and sine Fresnel integrals.

We can use the sectoral H plane gain curves on pages 226 and 228 for the corrugated horn. These curves are found from the equation.

$$\left(\frac{\lambda}{b}\text{Gain}\right) = \frac{4\pi R_x}{a}\left((C(u_x) - C(v_x))^2 + (S(u_x) - S(v_x))^2\right)$$

Therefore the gain of the corrugated horn is found from the curves with the formula.

$$\text{Gain} = \frac{1}{4\pi}\left(\frac{\lambda}{b} g_x\right)\left(\frac{\lambda}{a} g_y\right)$$

Example: Given a square corrugated horn with an aperture width of 4 wavelenghs and a slant radius of 5 wavelengths, find the gain.

From the curve on page 228 we find the gain.

$$g = \frac{1}{4\pi}(31.4)^2 = 78.5 \quad (18.9\text{ dB})$$

We can compare this with the gain of the square horn without corrugations.

$$g = \frac{\pi}{32}(31.4)(22.5) = 69.4 \quad (18.4\text{ dB})$$

The gain of the corrugated horn is greater than the horn without corrugations. At first it does not seem correct. The maximum gain is obtained from a uniformly excited aperture and this horn has more gain when the aperture has a cosine distribution. The thing we must remember is that both horns have phase error losses which have become the predominate factor. The maximum aperture phase deviation is 0.4 wavelengths. When the aperture phase deviation is greater than 0.25 wavelengths, the aperture field starts subtracting from the center fields. The cosine distribution outer edge fields subtract less from the center fields than does the uniform distribution. Hence the cosine distribution has more gain than the uniform amplitude distribution when the phase deviations from uniform are large. The optimum E plane sectoral horn occurs from a maximum phase deviation of 0.26 wavelengths.

On page 295 is a gain correction factor curve which can be used to find the gain of the rectangular corrugated horn and to design a horn to a particular gain.

$$\text{Gain} = 10\text{ Log}\left(\frac{4\pi a\, b}{\lambda^2}\right) - GF(S_x) - GF(S_y) \qquad S_x = \frac{a^2}{8\lambda R_x}$$

Example: Design a square corrugated horn at 6 GHz with 20 dB of gain.

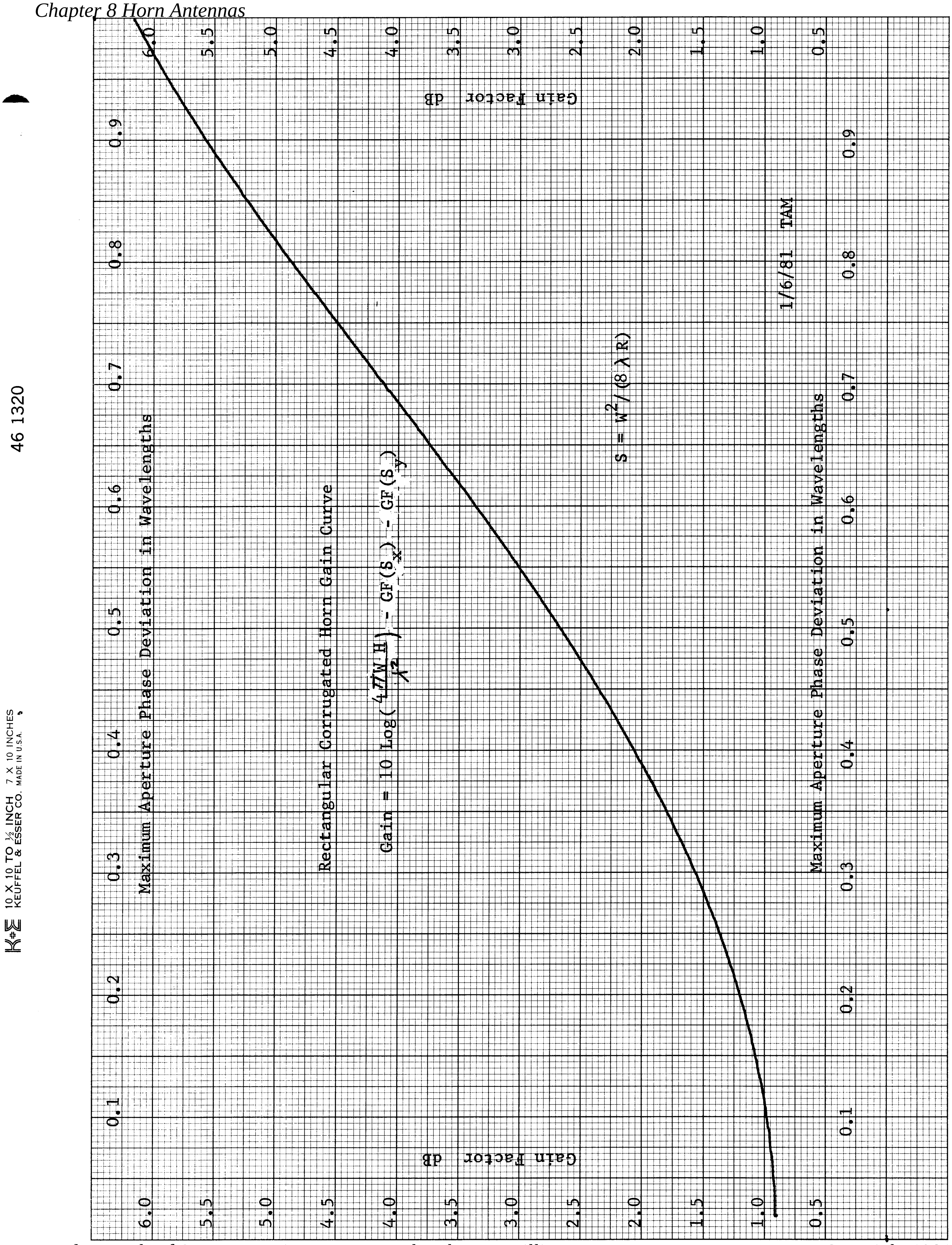
We will pick the optimum design which corresponds to $S = 0.4$

$$\frac{4\pi a^2}{\lambda^2} = 10^{(\text{gain} + 2\text{ GF})/20} \qquad \frac{a}{\lambda} = \frac{1}{2\sqrt{\pi}}\, 10^{(\text{gain} + 2\text{ GF})/20}$$

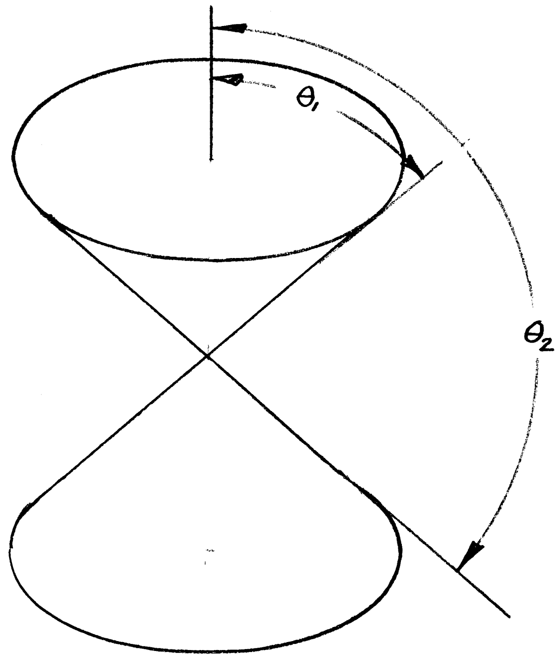
From the curve on page 295 we find the gain correction factor: 2.06 dB.

$$\frac{a}{\lambda} = \frac{1}{2\sqrt{\pi}}\, 10^{(20 + 4.12)/20} = 4.533 \qquad \frac{R}{\lambda} = \frac{(a/\lambda)^2}{8\text{ S}} = 6.422$$

At 6 GHz, the dimensions are $a = 8.917$ inches and $R = 12.63$ inches.


Rectangular Corrugated Horn Gain Curve
Gain = 10 Log(4πW H / λ2) - GF(Sx) - GF(Sy)
S = W2/(8 λ R)
1/6/81 TAM
Maximum Aperture Phase Deviation in Wavelengths
Gain Factor dB
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0 5.5 6.0
46 1320
K+E 10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.

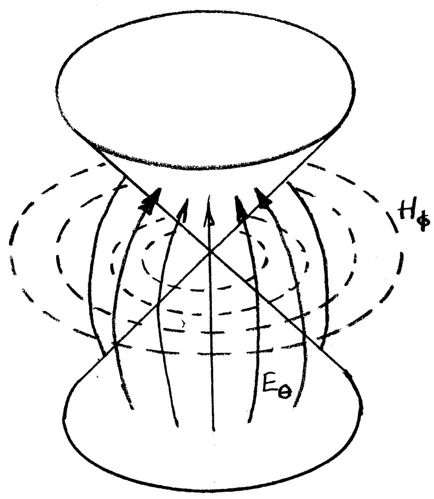
BICONICAL ANTENNA



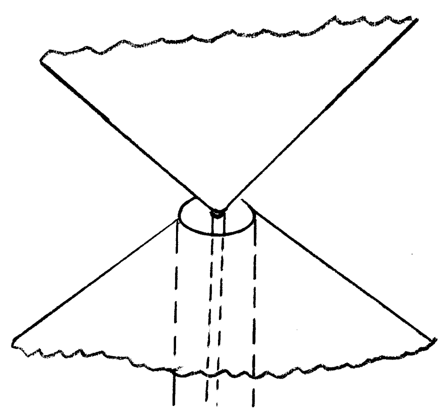
On page 24 we discussed briefly the biconical antenna. If we expand the Helmholtz equation in spherical coordinates, we would find a set of modes for a biconical waveguide. This process does not yield much useful information. The structure will support a TEM mode which is the primary mode. The TEM mode has the following fields.

$$E_\theta = \frac{E_o}{r \sin \theta} \qquad H_\phi = \frac{H_o}{r \sin \theta}$$

The field is propagating in the r direction. This mode gives vertical polarization similiar to a dipole with a narrower beamwidth. From symmetry we can see that all azimuth cuts are the same.



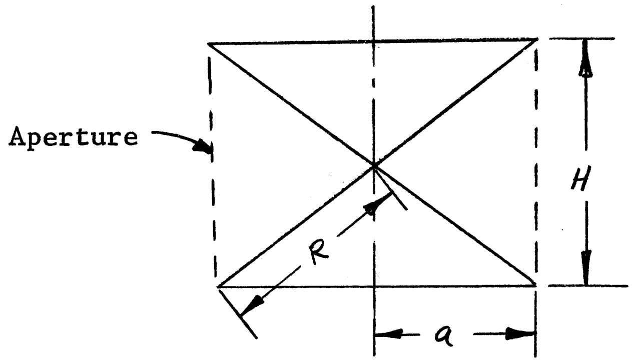
TEM Mode



Feed Region

The TEM mode is fed from a coax with a TEM mode as shown above and there is no low frequency cut off. The transition from coax to the biconical transmission line should be as smooth as possible with the upper cone connected to the center conductor and the outer shield of the coax connected to the lower cone.

The antenna can be analyzed by the Huygens source method and the gain at $\theta = 90^o$ will be found. It must be assumed that the aperture is large so that it can be assumed that the ratio of the electric field to the magnetic field is equal to the impedance of free space. Then we will analyze the antenna as a ring aperture. In the TEM mode the field will be considered constant in amplitude across the aperture with a quadratic phase error.



From the Huygens source the incremental electric field in the far field is given by

$$E_z\,dz = \frac{e^{-j\beta r}E_0\beta a}{4\pi r}\int_0^{2\pi}(1+\cos\phi')e^{j(\beta a\sin\theta\cos\phi')}\,d\phi'\,dz$$

Notice that the term $(1 + \text{Cos}\,\phi')$ is now included under the integral. The result of this integral is given in terms of Bessel functions.

$$E_z\,dz = \frac{e^{-j\beta r}E_0\beta a}{2r}\left[J_0(\beta a) + j\,J_1(\beta a\sin\theta)\right]dz$$

In the first higher order mode, TE_{01}, the electric field is horizontally polarized and since it must vanish at $Z = -H/2$ and $Z = H/2$ in the aperture, we will assume a cosine distribution in the aperture to satisfy the boundary conditions. The field should be given as a sum of spherical harmonics or Legendre polynomials of both kinds. The cosine approximation will give good results with a simpler integral. We can use Schelkunoff's integral results for horns to find the gain of the two different modes when we assume a quadratic phase error in the aperture.

TEM Mode (Vertical Polarization)

$$\text{Gain} = \frac{4\pi^2 a\,R}{H\lambda}\left(J_0^2(\beta a) + J_1^2(\beta a)\right)\left(C^2(w) + S^2(w)\right)$$

$$w = \frac{H}{\sqrt{2\lambda R}}$$

C(w) Cosine Fresnel Integral

S(w) Sine Fresnel Integral

TE_{01} Mode (Horizontal Polarization)

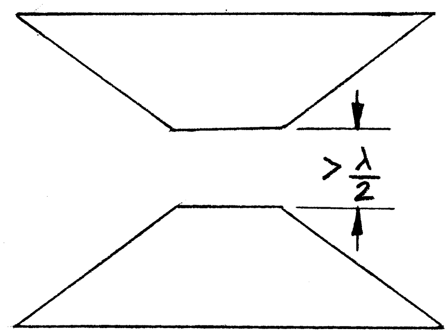
$$\text{Gain} = \frac{\pi^2 a\,R}{H\lambda}\left(J_0^2(\beta a) + J_1^2(\beta a)\right)\left((C(u) - C(v))^2 + (S(u) - S(v))^2\right)$$

$$u = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R}}{H} + \frac{H}{\sqrt{\lambda R}}\right) \qquad v = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\lambda R}}{H} - \frac{H}{\sqrt{\lambda R}}\right)$$

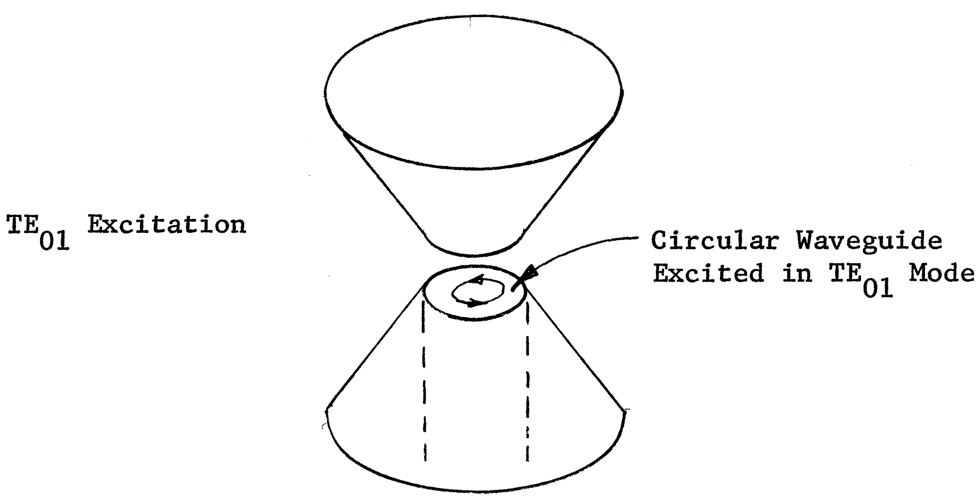
The TEM mode gain is plotted on page 299 versus the slant length and the total flare angle. On page 300 the gain is plotted similiar to horns. This plot shows that there is an optimum design for the biconical horn. we can use these curves to design the biconical horn.

The same two curves are plotted on pages 301 and 302 for the TE_{01} mode excitation of the horn. Like the TEM mode excitation, there is an optimum design for the TE_{01} mode as seen on page 302. Notice that the gain is much less in the TE_{01} mode than in the TEM mode.

The TEM mode can be easily excited by a coax input, but the TE_{01} mode is more difficult. The vertical distance between the cones must be greater than $\lambda/2$ before the mode can propagate between the cones.



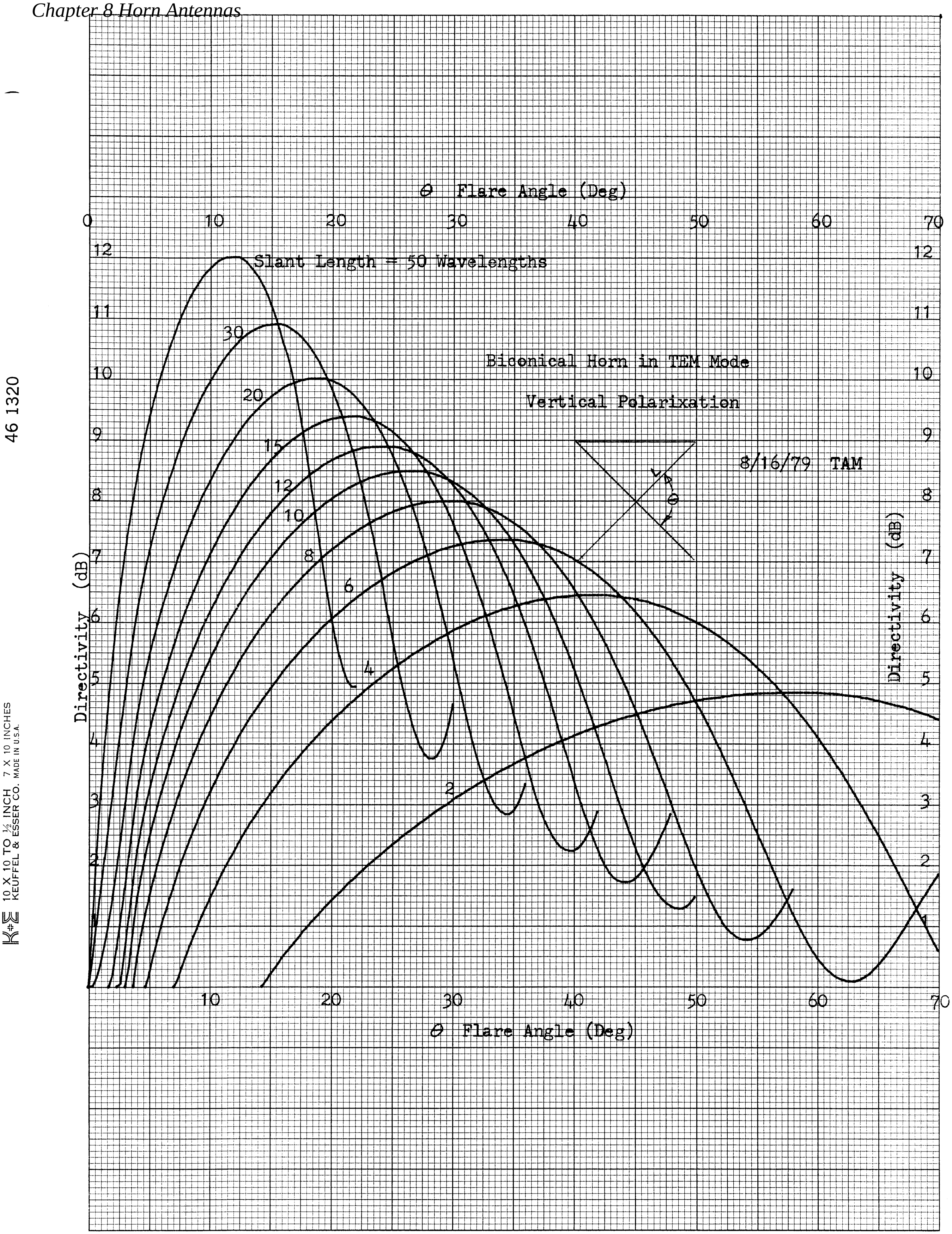
This is similiar to the rectangular waveguide. The best way to set up the horizontal mode is to feed the biconical horn with a circular waveguide excited in the TE_{01} mode.

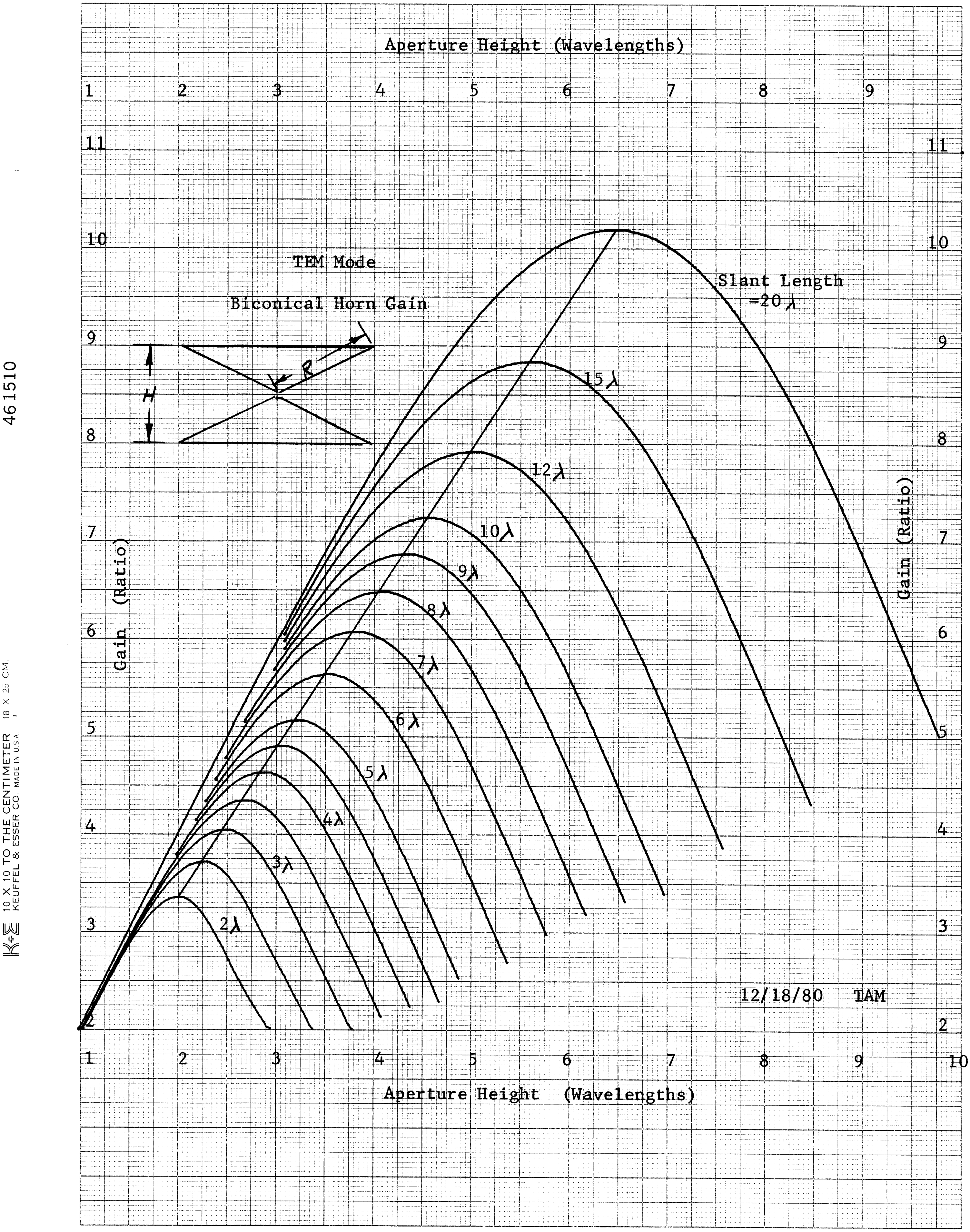


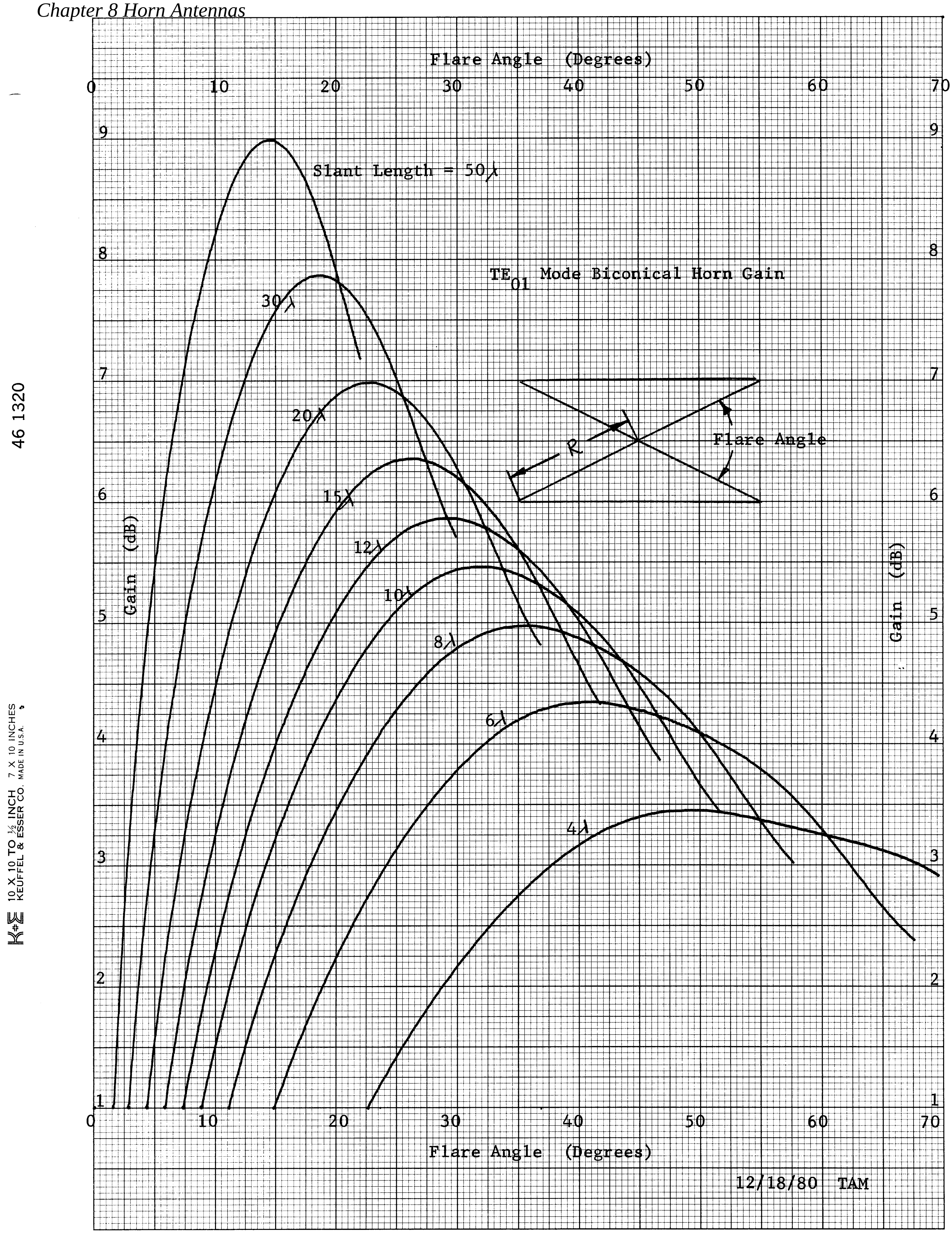
The fields in the biconical waveguide have circulating electric fields like the waveguide with the TE_{01} mode.

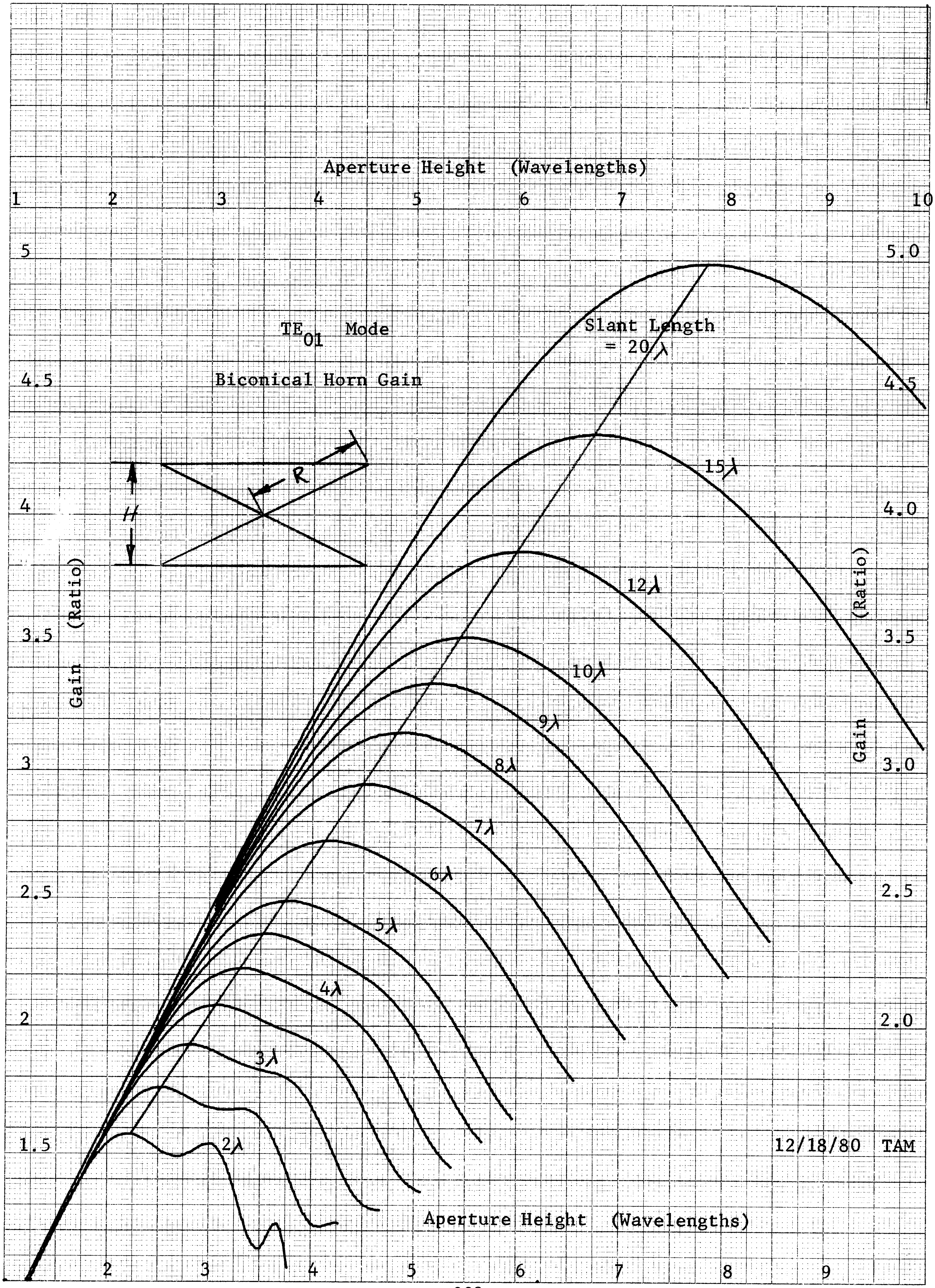
TEM Mode Impedance

The biconical transmission line between the two cones acts like any other TEM mode transmission line. For large biconical antennas the input


θ Flare Angle (Deg)
Slant Length = 50 Wavelengths
30
20
15
12
10
8
6
4
2
Biconical Horn in TEM Mode
Vertical Polarization
θ
8/16/79 TAM
Directivity (dB)
θ Flare Angle (Deg)
46 1320
10 X 10 TO ½ INCH 7 X 10 INCHES
KEUFFEL & ESSER CO. MADE IN U.S.A.


Aperture Height (Wavelengths)
TEM Mode
Biconical Horn Gain
H
R
Slant Length
=20λ
15λ
12λ
10λ
9λ
8λ
7λ
6λ
5λ
4λ
3λ
2λ
Gain (Ratio)
12/18/80 TAM
Aperture Height (Wavelengths)


Flare Angle (Degrees)
0
10
20
30
40
50
60
70
Slant Length = 50λ
TE$_{01}$ Mode Biconical Horn Gain
30λ
20λ
15λ
12λ
10λ
8λ
6λ
4λ
R
Flare Angle
Gain (dB)
1
2
3
4
5
6
7
8
9
12/18/80 TAM
46 1320
K+E 10 X 10 TO ½ INCH 7 X 10 INCHES KEUFFEL & ESSER CO. MADE IN U.S.A.



Aperture Height (Wavelengths)
1 2 3 4 5 6 7 8 9 10
TE01 Mode
Biconical Horn Gain
Slant Length
= 20λ
15λ
12λ
10λ
9λ
8λ
7λ
6λ
5λ
4λ
3λ
2λ
R
H
Gain (Ratio)
5 4.5 4 3.5 3 2.5 2 1.5
5.0 4.5 4.0 3.5 3.0 2.5 2.0
12/18/80 TAM
Aperture Height (Wavelengths)
2 3 4 5 6 7 8 9

impedance of the antenna is approximately the characteristic impedance of the biconical transmission line.

Evaluation of Biconical Transmission Line Impedance

The electric field between the cones is given by

$$E_\theta = \frac{E_o}{r \text{ Sin } \theta}$$

From the boundary conditions given on page 153, the charge density on the surface of the conductor is equal to the permittivity of the medium times the normal component of the electric field. E_θ is normal to the cones. The surface charge density is given as

$$\rho_s = \frac{\epsilon E_o}{r \text{ Sin } \theta_1}$$

on the upper cone. We want to find the differential charge with respect to the radius along the cone.

$$\frac{dQ}{dr} = \int_0^{2\pi} \frac{\epsilon E_0}{r \sin\theta_1}(r \sin\theta_1)\, d\phi = 2\pi\epsilon E_0$$

This is the charge per unit length along the cone.

The voltage between the cones can be found by integrating the electric field between the cones.

$$V = \int_{\theta_1}^{\theta_2} \frac{E_0}{r \sin\theta}\, r\, d\theta = E_0 \int_{\theta_1}^{\theta_2} \csc\theta\, d\theta$$

$$V = E_0\left[\ln\left(\tan\frac{\theta_2}{2}\right) - \ln\left(\tan\frac{\theta_1}{2}\right)\right]$$

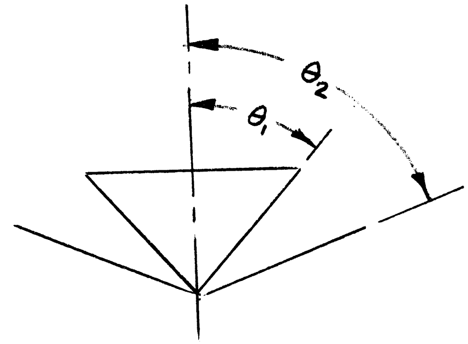
We use the voltage between the cones and the differential charge to find the differential capacitance between the cones.

$$\frac{dC}{dr} = \frac{dQ/dr}{V} = \frac{2\pi\epsilon}{\ln\left(\tan\frac{\theta_2}{2}\right) - \ln\left(\tan\frac{\theta_1}{2}\right)}$$

On page 70 there is an expression for the impedance from the capacitance per unit length and the velocity of light in the medium. We can use this to find the characteristic impedance.

$$Z = \frac{1}{vC} \qquad v = \frac{1}{\sqrt{\mu\epsilon}}$$

$$Z = \sqrt{\frac{\mu}{\epsilon}}\, \frac{\ln\left(\tan\frac{\theta_2}{2}\cot\frac{\theta_1}{2}\right)}{2\pi}$$



The characteristic impedance is given by

$$Z = \frac{\eta}{2\pi} \ln\left(\tan\frac{\theta_2}{2} \cot\frac{\theta_1}{2}\right)$$

This impedance is given in a nomograph on page 305. On page 306 is a plot of constant impedance contours. The straight line is for a complementary biconical transmission line.

Since antennas are designed for pattern and not impedance, the matching problem is worked after the proper pattern is obtained. It may mean feeding the biconical antenna through a transformer. When the biconical antenna is not large, then the input impedance will look like a terminated transmission line.

$$Z_{IN} = Z_0 \frac{Z_a \cos\beta r + j Z_0 \sin\beta r}{Z_0 \cos\beta r + j Z_a \sin\beta r}$$

Where Z_a is the effective radiation impedance referenced to the end of the biconical transmission line and Z_o is the characteristic impedance of the biconical transmission line.

PATTERN

In the TEM mode (Vertical Polarization) the θ component of the radiation intensity is given by the following integral.

$$U(\theta) = \left(\frac{E_0 \pi a}{\lambda}\right)^2 \frac{1}{\eta} \left(J_0^2(\beta a \sin\theta) + J_1^2(\beta a \sin\theta)\right) \sin\theta$$

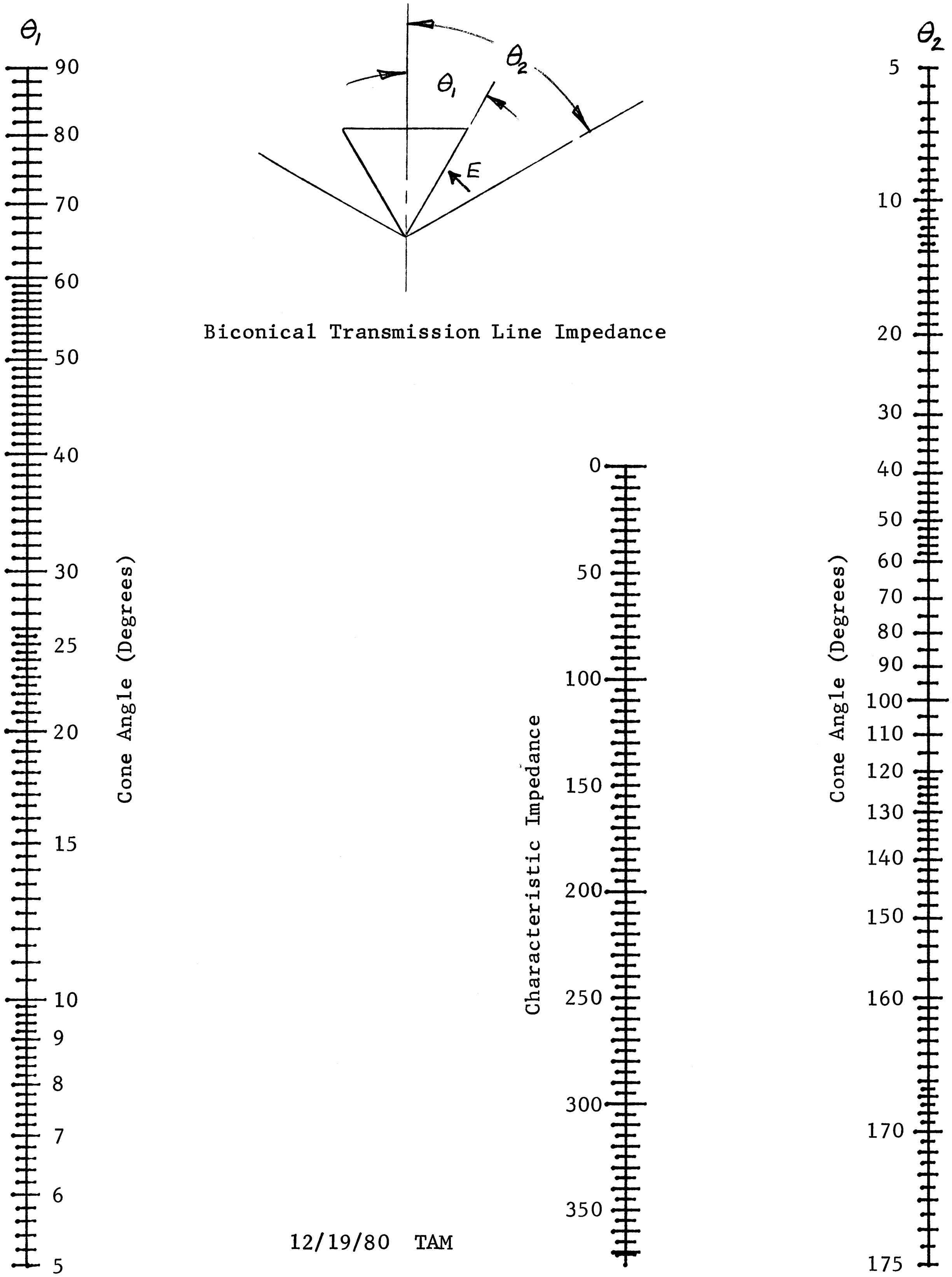
$$* \left| \int_{-H/2}^{H/2} e^{j\beta \cos\theta} e^{-j\frac{\pi z^2}{\lambda R}} dz \right|^2$$

For θ near 90^o and large diameter biconical antennas the factor

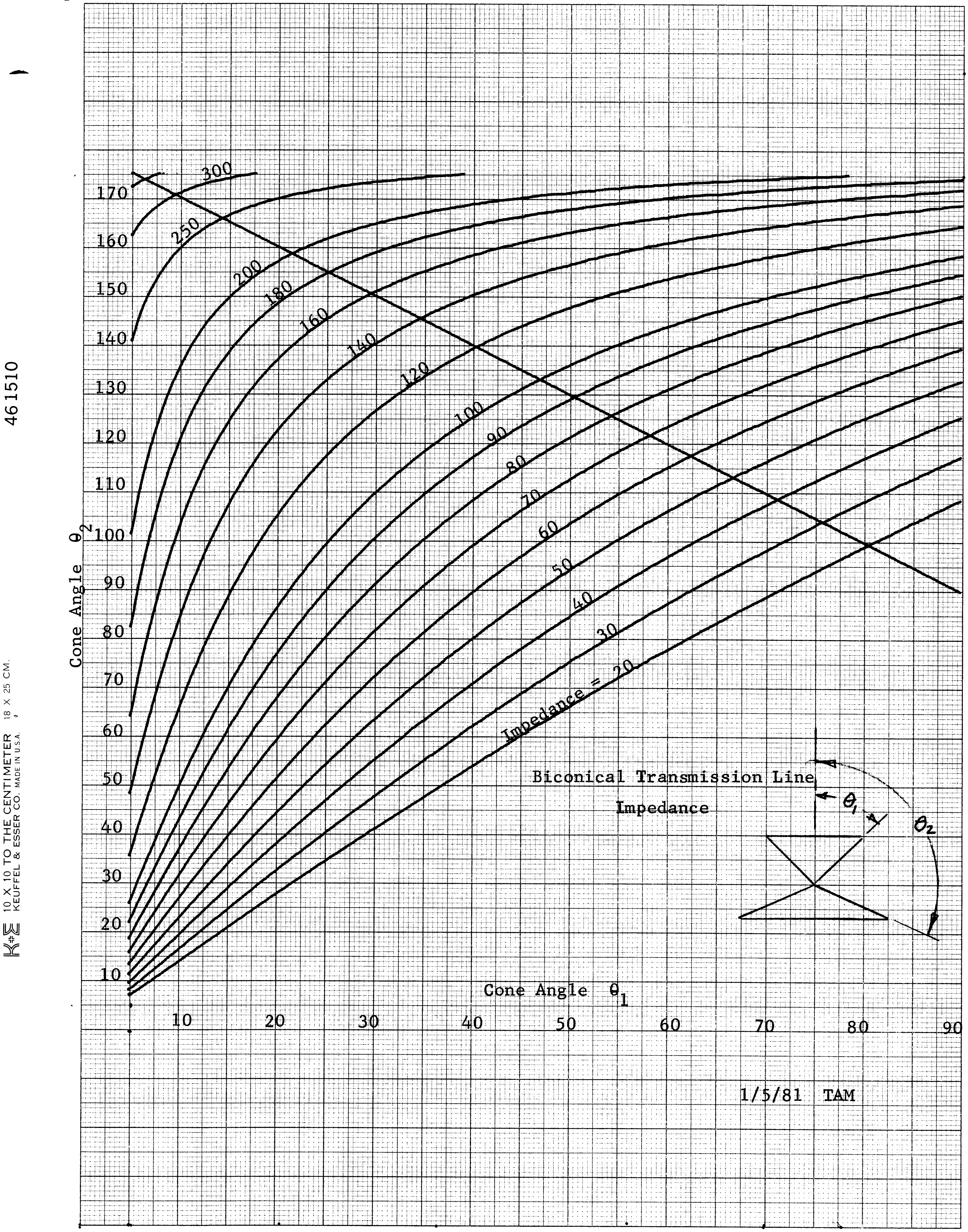
$$\left(J_0^2(\beta a \sin\theta) + J_1^2(\beta a \sin\theta)\right) \sin\theta$$

becomes an obliquity factor and the universal radiation pattern on page 220 of the E plane rectangular horn can be used with the TEM mode biconical horn where θ on the plot is measured from the maximum of the beam. Similiarly the H plane universal radiation pattern on page 221 can be used for the TE_{01} (Horizontal Polarization) mode.

Asymmetrical biconical antennas can be designed to point the beam up with limited success.



θ1
θ2
E
Biconical Transmission Line Impedance
Cone Angle (Degrees)
Characteristic Impedance
Cone Angle (Degrees)
90 80 70 60 50 40 30 25 20 15 10 9 8 7 6 5
0 50 100 150 200 250 300 350
5 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 175
12/19/80 TAM


Biconical Transmission Line
Impedance
Cone Angle θ_2
Cone Angle θ_1
Impedance = 20
30
40
50
60
70
80
90
100
120
140
160
180
200
250
300
θ_1
θ_2
1/5/81 TAM
10 X 10 TO THE CENTIMETER 18 X 25 CM.
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46 1510